ELECTRO MAGNETIC FIELDS
(3-0-0)
LECTURE NOTES
B. TECH
(II YEAR – III SEM)

Prepared by:

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Electromagnetic Fields (3-0-0) UPCEE303

Prerequisites:

1. Mathematics-I
2. Mathematics-II

Course Outcomes

At the end of the course, students will demonstrate the ability
1. To understand the basic laws of electromagnetism.
2. To obtain the electric and magnetic fields for simple configurations under static conditions.
3. To analyse time-varying electric and magnetic fields.
4. To understand Maxwell's equation in different forms and different media.
5. To understand the propagation of EM waves.

Module 1: (08 Hours)

Co-ordinate systems & Transformation: Cartesian co-ordinates, circular cylindrical coordinates, spherical coordinates. Vector Calculus: Differential length, Area & Volume, Line, surface and volume Integrals, Del operator, Gradient of a scalar, Divergence of a vector & Divergence theorem, Curl of a vector & Stoke's theorem, Laplacian of a scalar.

Module 2: (10 Hours)

Electrostatic Fields: Coulomb's Law, Electric Field Intensity, Electric Fields due to a point, line, surface and volume charge, Electric Flux Density, Gauss's Law- Maxwell's Equation, Applications of Gauss's Law, Electric Potential, Relationship between E and V- Maxwell's Equation and Electric Dipole & Flux Lines, Energy Density in Electrostatic Fields., Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions. Electrostatic boundary-value problems: Poisson's and Laplace's Equations, Uniqueness Theorem, General procedures for solving Poisson's and Laplace's equations, Capacitance.

Module 3: (06 Hours)

Magneto static Fields: Magnetic Field Intensity, Biot-Savart's Law, Ampere's circuit Law-Maxwell Equation, applications of Ampere's law, Magnetic Flux Density-Maxwell's equations. Maxwell's equation for static fields, Magnetic Scalar and Vector potentials. Magnetic Boundary Conditions.

Module 4: (10 Hours)

Electromagnetic Field and Wave propagation: Faraday's Law, Transformer & Motional Electromagnetic Forces, Displacement Current, Maxwell's Equation in Final forms, Time-Harmonic Field. Electromagnetic Wave Propagation: Wave Propagation in lossy Dielectrics, Plane Waves in loss less Dielectrics, Free space, Good conductors Power & Poynting vector.

TEXTBOOKS:

1. Matthew N. O. Sadiku, Principles of Electromagnetics, 6th Ed., Oxford Intl. Student Edition, 2014.

REFERENCE BOOKS:

1. C. R. Paul, K. W. Whites, S. A. Nasor, Introduction to Electromagnetic Fields, 3rd Ed, TMH.
2. W.H. Hyat, Electromagnetic Field Theory, 7th Ed, TMH.
3. A. Pramanik, "Electromagnetism - Theory and applications", PHI Learning Pvt. Ltd, New Delhi, 2009.
4. A. Pramanik, "Electromagnetism-Problems with solution", Prentice Hall India, 2012.
5. G.W. Carter, "The electromagnetic field in its engineering aspects", Longmans, 1954.
6. W.J. Duffin, "Electricity and Magnetism", McGraw Hill Publication, 1980.
7. W.J. Duffin, "Advanced Electricity and Magnetism", McGraw Hill, 1968.
8. E.G. Cullwick, "The Fundamentals of Electromagnetism", Cambridge University Press, 1966.
9. B. D. Popovic, "Introductory Engineering Electromagnetics", Addison- Wesley Educational Publishers, International Edition, 1971.
10. W. Hayt, "Engineering Electromagnetics", McGraw Hill Education, 2012.

MODULE-IV

1. Time Varying Fields
- Faraday’s Law
- Transformer & Motional Electromagnetic Forces
- Time Varying Electric and magnetic Fields
- Maxwell’s Equations
- Time Harmonic Fields

2. Electromagnetic Wave Propagation
- Wave Propagation in Lossy Dielectrics
- Plane Waves in Loss Less Dielectric, Free Space, Good Conductor
- Power and Poynting Vector

Chapter-1

Time Varying Fields

4.1 Introduction

ln the previous chapters we have studied the basic concepts in an electrostatic and magnetostatic fields. These fields can be considered as **time invariant** or static fields. In static electromagnetic fields, electric and magnetic fields are independent of each other. In this chapter, we shall concentrate on the time varying or dynamic fields. In **dynamic electromagnetic fields**, the electric and magnetic fields are **interdependent**. In general, static electric fields are produced by stationary electric charges. The static magnetic fields are produced due to the motion of the electric charges with uniform velocity or the magnetic charges. The time varying fields are produced due to the time varying currents.

4.2 Faraday's Law

According to Faraday's experiment, a static magnetic field cannot produce any current flow. But with a time varying field, an electromotive force (e.m.f.) induces which may drive a current in a closed path or circuit. This e.m.f. is nothing but a voltage that induces from changing magnetic fields or motion of the conductors in a magnetic field. Faraday discovered that the induced e.m.f is equal to the time rate of change of magnetic flux linking with the closed circuit.

Faraday's law can be stated as,

$$e = -N\frac{d\varphi}{dt} \text{ volts} \ldots\ldots\ldots\ldots\ldots \text{(1)}$$

where N = Number of turns in the circuit

e = Induced e.m.f.

Let us assume single turn circuit i.e. N = 1, then Faraday's law can be stated as,

$$e = -\frac{d\varphi}{dt} \text{ volts} \ldots\ldots\ldots\ldots\ldots \text{(2)}$$

The minus sign in equations (1) and (2) indicates that the direction of the induced e.m.f. is such that to produce a current which will produce a magnetic field which will oppose the original field. Thus according to Lenz's law, the induced e.m.f. acts to produce an opposing flux.

Let us consider Faraday's law. The induced e.m.f. is a scalar quantity measured in volts. Thus the induced e.m.f. is given by,

$$e = \oint_L \vec{E}.d\vec{L} \ldots\ldots\ldots\ldots \text{(3)}$$

The induced e.m.f. in equation (3) indicates a voltage about a closed path such that if any part of the path is changed, the e.m.f. will also change.

The magnetic flux passing through a specified area is given by,

$$\varphi = \int_S \vec{B} \cdot d\vec{S}$$

where $\vec{B}$ = Magnetic flux density

Using above result, equation (2) can be rewritten as,

$$e = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S} \text{ volts} \ldots\ldots (4)$$

From equations (3) and (4), we get,

$$e = \oint_L \vec{E} . d\vec{L} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S} \ldots\ldots\ldots\ldots (5)$$

There are two conditions for the induced e.m.f. as explained below.

i) The closed circuit in which e.m.f. is induced is stationary and the magnetic flux is sinusoidally varying with time. From equation (5) it is clear that the magnetic flux density is the only quantity varying with time. We can use partial derivative to define relationship as $\vec{B}$ may be changing with the co-ordinates as well as time. Hence we can write,

$$\oint_L \vec{E} . d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t} . d\vec{S} \ldots\ldots\ldots\ldots (6)$$

This is similar to transformer action and e.m.f. is called **transformer e.m.f.** Using Stoke's theorem, a line integral can be converted to the surface integral as

$$\oint_L \vec{E} . d\vec{L} = \int_S \left(\nabla \times \vec{E}\right) . d\vec{S} = -\int_S \frac{\partial \vec{B}}{\partial t} . d\vec{S} \ldots\ldots\ldots\ldots (7)$$

Assuming that both the surface integrals taken over identical surfaces.

$$\therefore \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ldots\ldots\ldots\ldots\ldots (8)$$

Equation (8) represents one of the Maxwell's equations. If $\vec{B}$ is not varying with time, then equations (6) and (8) give the results obtained previously in the electrostatics.

$$\oint_L \vec{E} . d\vec{L} = 0 \text{ and } \nabla \times \vec{E} = 0$$

ii) Secondly magnetic field is stationary, constant not varying with time while the closed circuit is revolved to get the relative motion between them. This action is similar to generator action, hence the induced e.m.f. is called **motional or generator e.m.f.**

Consider that a charge Q is moved in a magnetic field $\vec{B}$ at a velocity $\vec{v}$. Then the force on a charge is given by,

$$\vec{F} = Q\left(\vec{v} \, x \, \vec{B}\right) \ldots\ldots\ldots (9)$$

But the motional electric field intensity is defined as the force per unit charge. It is given by,

$$\vec{E}_m = \frac{\vec{F}}{Q} = \vec{v} \, x \, \vec{B} \ldots\ldots\ldots (10)$$

Thus the induced e.m.f. is given by,

$$\oint_L \vec{E}_m . d\vec{L} = \oint_L \left(\vec{v} \, x \, \vec{B}\right) . d\vec{L} \ldots\ldots\ldots\ldots (11)$$

Equation (11) represents total e.m.f. induced when a conductor is moved in a constant magnetic field.

iii) If in case, the magnetic flux density is also varying with time, then the induced e.m.f. is the combination of transformer e.m.f. and generator e.m.f. given by,

$$\oint_L \vec{E}_m . d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t} . d\vec{S} + \oint_L (\vec{v} \times \vec{B}) . d\vec{L} \ldots\ldots\ldots. (12)$$

Example 9.1 : *A conductor 1 cm in length is parallel to z-axis and rotates at radius of 25 cm at 1200 r.p.m. Find induced voltage, if the radial field is given by,*

$\bar{B} = 0.5\bar{a}_r$ T

Solution : In above case, the magnetic flux is constant while the path is rotating at 1200 r.p.m. Under such condition, the field intensity is given by,

$$\bar{E} = \bar{v} \times \bar{B}$$

where $\bar{v}$ = Linear velocity

In 1 minute there are 1200 revolutions which corresponds to 20 revolutions in one second. In one revolution distance travelled is $(2\pi r)$ meter. Hence in 20 revolutions the distance travelled in one second is $(40\pi r)$ meter. The conductor rotates in ϕ-direction. Hence linear velocity is given by,

$$\bar{v} = (40\pi r)\bar{a}_\phi$$
$$= 40\pi(25 \times 10^{-2})\bar{a}_\phi$$
$$= 31.416\ \bar{a}_\phi \text{ m/s}$$

Hence an electric field intensity is calculated as,

$$\bar{E} = [31.416\bar{a}_\phi] \times [0.5\bar{a}_r]$$
$$= 15.708\ (-\bar{a}_z) \qquad \ldots\ \bar{a}_\phi \times \bar{a}_r = -\bar{a}_z$$

Induced voltage is given by,

$$e = \oint \bar{E} \cdot d\bar{L}$$

Now $d\bar{L} = (dz)\bar{a}_z$ as conductor is parallel to z-axis.

$$e = \int_{z=0}^{0.01} 15.708(-\bar{a}_z) \cdot (dz)\,\bar{a}_z$$
$$= -15.708[z]_0^{0.01} = -157.08 \text{ mV}$$

Negative sign indicates upper end of the conductor is positive while lower end is negative. Thus the magnitude of the induced voltage is 157.08 mV.

4.3 Displacement Current

Consider Maxwell's curl equation for magnetic fields (Ampere's Circuit Law) for time-varying conditions.

For static EM fields, we recall that

$$\nabla \times \vec{H} = \vec{J} \ldots\ldots\ldots (1)$$

But the divergence of the curl of any vector field is identically zero. Hence,

$$\nabla . (\nabla \times \vec{H}) = 0 = \nabla . J \ldots\ldots\ldots (2)$$

But the continuity of current equation requires that

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \ldots\ldots\ldots (3)$$

From equation (3) it is clear that when $\nabla.\vec{J} = 0$, then only equation (2) becomes true. Thus equations (2) and (3) are not compatible for time varying fields. We must modify equation (1) by adding one unknown term say $\vec{J}_d$.Then equation (1) becomes,

$$\nabla \times \vec{H} = \vec{J} + \vec{J}_d \ldots\ldots\ldots (4)$$

Again taking divergence on both the sides

$$\nabla.\left(\nabla \times \vec{H}\right) = \nabla.\vec{J} + \nabla.\vec{J}_d = 0$$

As $\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$, to get correct conditions we must write,

$$\nabla.\vec{J}_d = \frac{\partial \rho_v}{\partial t}$$

But according to Gauss's law,

$$\rho_v = \nabla.\vec{D}$$

Thus replacing ρ_v by $\nabla.\vec{D}$

$$\nabla.\vec{J}_d = \frac{\partial}{\partial t}(\nabla.\vec{D})$$

$$= \nabla.\frac{\partial \vec{D}}{\partial t}$$

Comparing two sides of the equation,

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} \ldots\ldots\ldots\ldots (5)$$

Now we can write Ampere's circuital law in point form as,

$$\nabla \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t} \ldots\ldots\ldots (6)$$

The first term in equation (6) is conduction current density denoted by $\vec{J}_c$. Here attaching subscript C indicates that the current is due to the moving charges. The second term in equation (6) represents current density expressed in ampere per square meter. As this quantity is obtained from time varying electric flux density. This is also called displacement density. Thus this is called **displacement current density** denoted by $\vec{J}_d$. With these definitions we can write equation (6) as,

$$\nabla \times \vec{H} = \vec{J}_c + \vec{J}_d \ldots\ldots\ldots (7)$$

4.4 Maxwell's Equations for Time Varying Fields

Maxwell's equations are nothing but a set of four expressions derived from Ampere's circuit law, Faraday's law, Gauss's law for electric field and Gauss's law for magnetic field.

These four expressions can be written in following forms

i) Point or differential form, ii) Integral form

Let us summarize the Maxwell's equations for time-varying electric and magnetic fields.

Maxwell's equations in differential or point form	
1. $\nabla.\vec{D} = \rho_v$	Gauss's law
2. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Conservation of electric field (Faraday's Law)
3. $\nabla \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$	Ampere's circuital law
4. $\nabla.\vec{B} = 0$	Single magnetic pole cannot exist i.e. conservation of magnetic flux.

Table 3.1

The Maxwell's equations in integral form an be summarized as,

Maxwell's equations in integral form
1. $\oint_S \vec{D}.d\vec{S} = \int_v \rho_v \, dv = Q$
2. $\oint_L \vec{E}.d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$
3. $\oint_L \vec{H}.d\vec{L} = \int_S \vec{J}.d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t}.d\vec{S}$
4. $\oint_S \vec{B}.d\vec{S} = 0$

Table 3.2

Example 9.5 : *If the magnetic field $\bar{H} = [3x\cos\beta + 6y\sin\alpha]\bar{a}_z$, find current density $\bar{J}$ if fields are invariant with time.*

Solution : The point form of Maxwell's second equation is,

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

But as fields are time invariant, we can write,

$$\frac{\partial \bar{D}}{\partial t} = 0$$

$$\therefore \quad \nabla \times \bar{H} = \bar{J}$$

$$\therefore \quad \bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (3x\cos\beta + 6y\sin\alpha) \end{vmatrix}$$

$$\therefore \quad \bar{J} = \frac{\partial}{\partial y}[3x\cos\beta + 6y\sin\alpha]\,\bar{a}_x - \frac{\partial}{\partial x}[3x\cos\beta + 6y\sin\alpha]\,\bar{a}_y$$

$$\therefore \quad \bar{J} = 6\sin\alpha\,\bar{a}_x - 3\cos\beta\,\bar{a}_y \quad A/m^2$$

4.5 Time Harmonic Fields

A **time-harmonic field** is one that varies periodically or sinusoidally with time. Sinusoids are easily expressed in phasors, which are more convenient to work with.

A phasor is a complex number that contains the amplitude and the phase of a sinusoidal oscillation. In general, any complex number z can be written as,

$$z = a + jb = r\angle\theta^0 \text{ .... (1)}$$

$$\text{or } m = re^{j\theta} = r(cos\theta + j\, sin\theta) \text{ .... (2)}$$

In equation (1) and (2), a and b are the real and imaginary parts of complex number m. The symbol j represents complex operator. Its value is $\sqrt{-1}$. The magnitude of m is given by,

$$r = |m| = \sqrt{a^2 + b^2} \text{ .... (3)}$$

The phase angle is given by,

$$\theta = \tan^{-1}\frac{b}{a} \text{ .... (4)}$$

From above discussion, it is clear that any phasor can be represented in rectangular as well as polar form represented by equations (1) to (4). Note that the phasor representation is applicable only to the sinusoidal signals. Any sinusoidal signal can be defined with the help of three parameters namely amplitude, frequency and phase. Let the applied electric field is given by,

$$E = E_m\, cos\,(\omega t + \phi)$$

where E_m = Amplitude, ωt = Angular frequency and ϕ = Phase angle

According to Euler's identity, $e^{j\theta} = cos\theta + j\, sin\theta$

Thus the real and imaginary parts of $E_m e^{j\theta}$ (where $\theta = \omega t + \phi$) are given by,

$$\text{Re}(E_m e^{j\theta}) = E_m\, cos\,(\omega t + \phi) \text{ ..... (5)}$$

$$\text{and Im}\,(E_m e^{j\theta}) = E_m\, sin\,(\omega t + \phi) \text{ ....(6)}$$

Hence we can write,

$$E = \text{Re}\big(E_m e^{j\theta}\big) = \text{Re}(E_m e^{j\omega t} e^{j\phi}) \text{ .... (7)}$$

The complex term $E_m e^{j\phi}$ is called phasor. Generally, it is represented by attaching suffix s to the quantity of concern, such as E_s.

A phasor may be either scalar or vector. Let the vector $\vec{M}$ is time varying field which varies with respect of x, y, z and t. Then the phasor form of $\vec{M}$ is obtained by dropping the time factor. Let it be $\vec{M}_s$ which depends only on x, y and z. Then the two quantities are related to each other by the relation,

$$\vec{M} = Re(\vec{M}_s\, e^{j\omega t}) \text{ .... (8)}$$

Differentiating $\vec{M}$ with respect to t partially,

$$\frac{\partial \vec{M}}{\partial t} = \frac{\partial}{\partial t} Re(\vec{M}_s\, e^{j\omega t})$$

$$\frac{\partial \vec{M}}{\partial t} = Re(j\omega \vec{M}_s\, e^{j\omega t}) \ldots\ldots\ldots\ldots (9)$$

Similarly, we can write,

$$\int \vec{M}\, \partial t = \int Re(\vec{M}_s\, e^{j\omega t})\, \partial t$$

$$\int \vec{M}\, \partial t = Re\left(\frac{\vec{M}_s}{j\omega}\, e^{j\omega t}\right) \ldots\ldots\ldots\ldots (10)$$

Key Point: From equations (9) and (10) it is clear that, differentiating and integrating the quantity with respect to time is equivalent in multiplying and dividing the phasor of that quantity by factor $j\omega$ respectively.

Examples with Solutions

Example 9.7 : *A rectangular conducting loop with a resistance of 0.2 Ω rotates at 500 r.p.m. The vertical conductor at $r_1 = 0.03$ m is in the field $\bar{B}_1 = 0.25\, \bar{a}_r$ T and other conductor is at $r_2 = 0.05$ m and in the field $\bar{B}_2 = 0.8\, \bar{a}_r$ T. Find current flowing in the loop.*

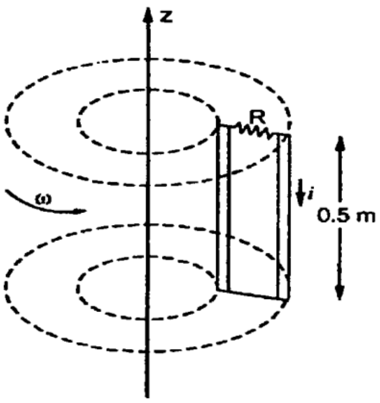

Fig. 9.4

Solution : The inner conductor which is at $r_1 = 0.03$ m rotates at 500 r.p.m. Thus inner conductor rotates with $\frac{500}{60}$ revolutions per second. As in one second, the distance covered is $(2\pi r)$ meter, for the inner conductor the distance covered is $\left(\frac{500}{60}\right)(2\pi r)$ meters.

Then the linear velocity for inner conductor is given by,

$$\bar{v}_1 = \left(\frac{500}{60}\right)(2\pi)(0.03)\, \bar{a}_\phi \quad m/s$$

$$= 1.5707\, \bar{a}_\phi \ m/s$$

Similarly for outer conductor, linear velocity is given by,

$$\bar{v}_2 = \left(\frac{500}{60}\right)(2\pi)(0.05)\, \bar{a}_\phi \quad m/s$$

$$= 2.6179\, \bar{a}_\phi \ m/s$$

Here $\bar{B}$ is not varying with time, it is constant in $\bar{a}_r$ direction. Thus under such condition, the induced e.m.f. is given by,

$$\text{e.m.f.} = \int \bar{E} \cdot d\bar{L}$$

where $\qquad \bar{E} = \bar{v} \times \bar{B}$

For inner conductor,

$$\bar{E}_1 = \bar{v}_1 \times \bar{B}_1$$
$$= [1.5707\ \bar{a}_\phi] \times [0.25\ \bar{a}_r]$$
$$= -0.3926\ \bar{a}_z \qquad \text{.......}\ \bar{a}_\phi \times \bar{a}_r = -\bar{a}_z$$

Both the conductors are vertical. Let us assume that length of each conductor be 0.5 m.

$$\therefore \quad d\bar{L}_1 = dz\,\bar{a}_z$$

$$\therefore \quad \text{e.m.f.}_1 = \int \bar{E}_1 \cdot d\bar{L}_1 = \int_{z=0}^{0.5\,m} (-0.3926\,\bar{a}_z) \cdot (dz\,\bar{a}_z)$$
$$= -0.3925\,[z]_0^{0.5}$$
$$= -0.1963\ \text{V}$$

For outer conductor,

$$\bar{E}_2 = \bar{v}_2 \times \bar{B}_2$$
$$= [2.6179\ \bar{a}_\phi] \times [0.8\ \bar{a}_r]$$
$$= -2.09432\ \bar{a}_z \qquad \text{.......}\ \bar{a}_\phi \times \bar{a}_r = -\bar{a}_z$$

$$\therefore \quad \text{e.m.f.}_2 = \int \bar{E}_1 \cdot d\bar{L}_2$$
$$= \int_{z=0}^{0.5\,m} (-2.09432)\ \bar{a}_z \cdot dz\,\bar{a}_z$$
$$= -2.09432\,[z]_0^{0.5}$$
$$= -1.04716\ \text{V}$$

Hence current in the loop is given by,

$$i = \frac{\text{e.m.f.}_1 - \text{e.m.f.}_2}{R} = \frac{-0.1963-(-1.04716)}{0.2}$$

$$\therefore \quad i = \frac{0.85086}{0.2} = 4.2543\ \text{A}$$

Example 9.8 : *The circular loop conductor having a radius of 0.15 m is placed in X-Y plane. This loop consists of a resistance of 20 Ω as shown in the Fig. 9.5. If the magnetic flux density is*

$$\bar{B} = 0.5\sin 10^3 t\ \bar{a}_z\ \text{T}$$

Find current flowing through this loop.

Solution : The circular loop conductor is in X-Y plane. $\bar{B}$ is in $\bar{a}_z$ direction which is perpendicular to X-Y plane.

Hence, we can write,

$$d\bar{S} = (r\,dr\,d\phi)\bar{a}_z$$

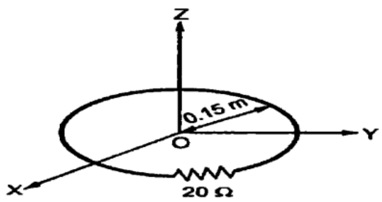

Fig. 9.5

Total flux is given by,

$$\Phi = \int_S \bar{B} \cdot d\bar{S}$$
$$= \int_{\phi=0}^{2\pi} \int_{r=0}^{0.15} [(0.5\sin 10^3 t)\bar{a}_z] \cdot [(r\,dr\,d\phi)\bar{a}_z]$$
$$= (0.5\sin 10^3 t)[\phi]_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{0.15}$$
$$= (0.5\sin 10^3 t)\,[2\pi]\left[\frac{(0.15)^2}{2}\right]$$
$$= 35.3429\sin 10^3 t \quad \text{mWb}$$

Now induced e.m.f. is given by,

$$e = -\frac{d\Phi}{dt}$$

$$= -\frac{d}{dt}\left[35.3429\times10^{-3}\sin 10^3 t\right]$$

$$= -\left(35.3429\times10^{-3}\right)\left(10^3\right)\cos 10^3 t$$

$$= -35.3429 \cos 10^3 t \ \ V$$

Hence current in the conductor is given by,

$$i = \frac{e}{R} = \frac{-35.3429\cos 10^3 t}{20}$$

$$\therefore \qquad i = \mathbf{-1.7671 \cos 10^3 t \ \ A}$$

Example 9.9 : *An area of 0.65 m² in the plane z = 0 encloses a filamentary conductor. Find the induced voltage if,*

$$\bar{B} = 0.05\cos 10^3 t\left(\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right) \text{ Tesla.}$$

Solution : Here filamentary conductor is fixed and it is placed in z = 0 plane. It encloses area of 0.65 m².

$$\therefore \qquad d\bar{S} = dS\,\bar{a}_z$$

Induced e.m.f. according to Faraday's law is given by,

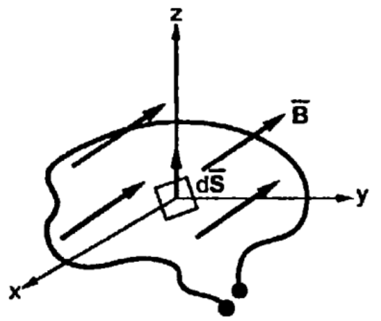

Fig. 9.6

$$e = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}$$

$$= -\int_S \frac{\partial}{\partial t}\left[0.05\cos 10^3 t\left(\frac{\bar{a}_y + \bar{a}_z}{\sqrt{2}}\right)\right]\cdot(dS\,\bar{a}_z)$$

$$= -\int_S \frac{0.05\left(10^3\right)\left(-\sin 10^3 t\right)}{\sqrt{2}}\,dS \qquad \ldots \bar{a}_y\cdot\bar{a}_z = 0,\ \ \bar{a}_z\cdot\bar{a}_z = 1$$

$$= +35.355 \sin 10^3 t\left[\int_S dS\right]$$

But $\int_S dS$ is given as 0.65 m².

$$\therefore \qquad e = 35.355\sin 10^3 t\ (0.65)$$

$$\therefore \qquad = \mathbf{22.98 \sin 10^3 t \ \ V}$$

Example 9.10 : *A conducting cylinder of radius 7 cm and height 50 cm rotates at 600 r.p.m. in a radial field $\bar{B} = 0.10\,\bar{a}_r$ T. Sliding contacts at the top and bottom are used to connect a voltmeter as shown in the Fig. 9.7. Calculate induced voltage.*

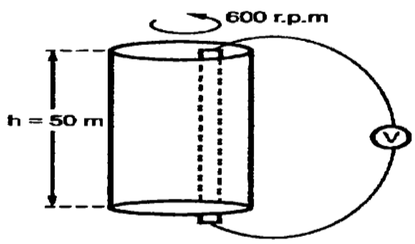

Fig. 9.7

Solution : A conducting cylinder rotates in the direction as shown in the Fig. 9.7. It rotates at 600 r.p.m. Means in 1 sec there are 10 revolutions. The radius of the cylinder is 0.07 m. In 1 revolution, the distance travelled by the cylinder is $(2\pi r)$ m i.e. $(2\times\pi\times0.07)$ m. Hence in 10 revolutions, it travels $(2\times\pi\times0.07\times10)$ m distance. So the linear velocity is given by,

$$\bar{v} = (2\times\pi\times0.07\times10)\bar{a}_\phi \text{ m/s}$$

$$= 4.398\ \bar{a}_\phi \text{ m/s}$$

The electric field intensity is given by,

$$\bar{E} = \bar{v}\times\bar{B}$$

$$= (4.398\ \bar{a}_\phi)\times(0.20)\ \bar{a}_r$$

$$= 0.8796\ (-\bar{a}_z) \qquad \ldots\ \bar{a}_\phi\times\bar{a}_r = -\bar{a}_z$$

Here field is not varying with time. The cylindrical conductor is rotating. Each vertical element of it on the curved surface cuts same flux and thus the induced voltage is same. As these elements are as if in parallel, the e.m.f. induced in one element is same as that total e.m.f.

$$\therefore \qquad e = \int \bar{E}\cdot d\bar{L}$$

$$= \int_{z=0}^{0.5} 0.8796\ (-\bar{a}_z)\cdot(dz\,\bar{a}_z)$$

$$= -0.8796\ [z]_0^{0.5} \qquad \ldots\ \bar{a}_z\cdot\bar{a}_z = 1$$

$$= -0.4398 \text{ V}$$

Example 9.11 : *Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2\times10^{-4}$ ℧/m and $\varepsilon_r = 81$.*

Solution : The ratio of amplitudes of the two current densities is given as 1, so we can write,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = 1$$

i.e.
$$\omega = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_0\,\varepsilon_r}$$

$$\therefore \qquad \omega = \frac{2\times10^{-4}}{(8.854\times10^{-12})(81)} = 0.2788\times10^6 \text{ rad/sec}$$

But
$$\omega = 2\pi f$$

$$\therefore \qquad f = \frac{\omega}{2\pi} = \frac{0.2788\times10^6}{2\pi} = 44.372 \text{ kHz}$$

Hence, the frequency at which the ratio of amplitudes of conduction and displacement current density is unity, is 44.372 kHz.

Example 9.31 : *A No 10 copper wire carries a conduction current of 1 amp at 60 Hz. Calculate the displacement current in the wire. For copper assume,*

$$\varepsilon = \varepsilon_0 = \frac{1}{36\times\pi\times 10^9} \text{ F/m} = 8.854 \times 10^{-12} \text{ F/m}$$

$$\mu = \mu_0 = 4\pi\times10^{-7} \text{ H/m}$$

$$\sigma = 5.8 \times 10^7 \text{ ℧/m}$$

(UPTU : 2005-06)

Solution : By definition,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = \frac{5.8\times10^7}{2\times\pi\times60\times8.854\times10^{-12}} = 1.7376 \times 10^{16}$$

But $\quad |\bar{J}_C| = \dfrac{i_C}{A}$ and $|\bar{J}_D| = \dfrac{i_D}{A}$

$$\therefore \quad \frac{i_C / A}{i_D / A} = 1.7376 \times 10^{16}$$

$$\therefore \quad i_D = \frac{i_C}{1.7376\times10^{16}} = \frac{1}{1.7376\times10^{16}} = 0.05755 \times 10^{-15} \text{ A}$$

Example 9.32 : *Consider a loop as shown in the Fig. 9.12. If $\bar{B} = 0.5\ \bar{a}_z$ Wb/m², $R = 20\Omega$, l = 10 cm and rod is moving with constant velocity of $8\ \bar{a}_x$ m/sec, find*

i) the induced e.m.f. in the rod,

ii) the current through the resistance

iii) the motional force on the rod,

iv) the power dissipated by the resistance

(UPTU : 2006-07, 10 Marks)

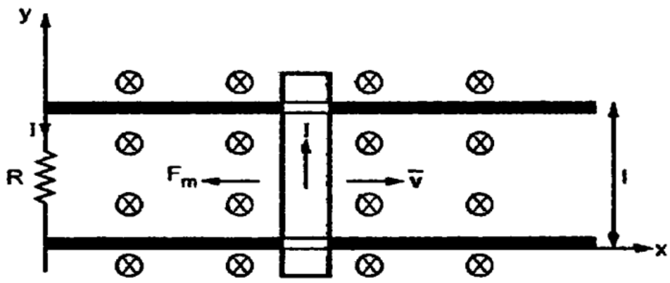

Fig. 9.12

Solution : i) The induced motional e.m.f. is given by,

$$e = \int(\bar{v}\times\bar{B})\bullet d\bar{L} = \int_{l=10\times10^{-2}}^{0}(8\ \bar{a}_x \times 0.5\ \bar{a}_z)\bullet(dy\ \bar{a}_y)$$

$$= 4\int_{10\times10^{-2}}^{0}(-\bar{a}_y)\bullet(dy\ \bar{a}_y) = -4\int_{10\times10^{-2}}^{0} dy$$

$$= -4\ [0 - 10 \times 10^{-12}]$$

$$= 0.4 \text{ V}$$

ii) The current through resistor $= \dfrac{V_{induced}}{R} = \dfrac{e}{R} = \dfrac{0.4}{20} = 20$ mA

iii) The motional force on the bar is given by,

$$F = BIl = (0.5)\ (20 \times 10^{-3})\ (10 \times 10^{-2}) = 1 \times 10^{-3}\text{N} = 1 \text{ mN}$$

iv) The power dissipated by resistance is given by,

$$P_D = I^2R = (20 \times 10^{-3})^2 \times 20 = 8 \times 10^{-3} \text{ W} = 8 \text{ mW}$$

Electromagnetic (EM) Wave Propagation

4.6 Introduction

In general, **waves** are means of transporting energy or information. Typical examples of EM waves include radio waves, TV signals, radar beams and light rays. Our major goal is to solve Maxwell's equations and describe EM wave motion in the following media:

1. Free space $(\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0)$
2. Lossless dielectrics $(\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0, or\ \sigma \ll \omega\varepsilon)$
3. Lossy dielectrics $(\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_0, \mu = \mu_r\mu_0)$
4. Good Conductors $(\sigma \approx 0, \varepsilon = \varepsilon_0, \mu = \mu_r\mu_0, or\ \sigma \gg \omega\varepsilon)$

Where ω is the angular frequency of the wave. Lossy dielectrics, is the most general case and wave equation is derived for this case and other cases are derived from it as special cases by changing the values of $\sigma, \varepsilon\ and\ \mu$.

4.6.1 Wave Propagation in Lossy Dielectrics

A **lossy dielectric** is a medium in which an EM wave, as it propagates, loses power owing to imperfect dielectric with $\sigma \neq 0$.

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free $(\rho_v = 0)$. Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's become

$$\nabla . \vec{E}_s = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\nabla . \vec{H}_s = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\nabla \times \vec{E}_s = -j\omega\mu\vec{H}_s \ldots\ldots\ldots\ldots (3)$$

$$\nabla \times \vec{H}_s = (\sigma + j\omega\varepsilon)\vec{E}_s \ldots\ldots\ldots (4)$$

Taking the curl of both sides of equation (3) gives

$$\nabla \times \nabla \times \vec{E}_s = -j\omega\mu(\nabla \times \vec{H}_s) \ldots\ldots (5)$$

Applying the vector identity

$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla . \vec{A}\right) - \nabla^2\vec{A} \ldots\ldots\ldots\ldots (6)$$

To the left-hand side of equation (5) and invoking equations (1) and (4), we obtain

$$\nabla\left(\nabla . \vec{E}_s\right) - \nabla^2\vec{E}_s = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}_s$$

$$\text{Or} \quad \nabla^2\vec{E}_s - \gamma^2\vec{E}_s = 0 \ldots\ldots (7)$$

$$\text{Where} \quad \gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \ldots\ldots\ldots (8)$$

And γ is called the **propagation constant** of the medium. By a similar procedure, it can be shown that for the $\vec{H}$ field,

$$\nabla^2\vec{H}_s - \gamma^2\vec{H}_s = 0 \ldots\ldots\ldots (9)$$

Equations (7) and (9) are known as homogeneous vector **Helmholtz's equations** or simply **wave equations**.

Since γ in equations (7) to (9) is a complex quantity, we may let

$$\gamma = \alpha + j\beta \text{ …… (10)}$$

We obtain α *and* β from equations (8) and (10) by noting that

$$-Re\,\gamma^2 = \beta^2 - \alpha^2 = \omega^2\mu\varepsilon \text{ …….. (11)}$$

And
$$|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2} \text{…….. (12)}$$

From equations (11) and (12), we obtain

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]} \text{……. (13)}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]} \text{……. (14)}$$

Without loss of generality, if we assume that the wave propagates along $+\vec{a}_z$ and that $\vec{E}_s$ has only an x-component, then

$$\vec{E}_s = E_{xs}(z)\vec{a}_x \text{ ……… (15)}$$

Substituting this into equation (7) yields

$$(\nabla^2 - \gamma^2)E_{xs}(z) = 0 \text{ ……… (16)}$$

Hence
$$\frac{\partial^2 E_{xs}(z)}{\partial x^2} + \frac{\partial^2 E_{xs}(z)}{\partial y^2} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

Or
$$\left[\frac{d^2}{dz^2} - \gamma^2\right]E_{xs}(z) = 0 \text{………. (17)}$$

This is a scalar wave equation, a linear homogeneous differential equation, with solution

$$E_{xs}(z) = E_0 e^{-\gamma z} + E_0{}' e^{\gamma z} \text{………. (18)}$$

Where E_0 *and* $E_0{}'$ are constants. Since $e^{\gamma z}$ denotes a wave travelling along $-\vec{a}_z$ whereas we assume wave propagation along $\vec{a}_z$, $E_0{}' = 0$. Inserting the time factor $e^{j\omega t}$ into equation (18) and using equation (10), we obtain

$$\vec{E}(z,t) = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\vec{a}_x \text{…….. (19)}$$

By following the similar steps, we obtain

$$\vec{H}(z,t) = Re\left(H_0 e^{-\alpha z} \mathrm{e}^{j(\omega t - \beta z)}\,\vec{a}_y\right) \text{….. (20)}$$

Where
$$H_0 = \frac{E_0}{\eta} \text{ …………………..(21)}$$

And η is a complex quantity known as the intrinsic impedance, in ohms, of the medium.

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta|\angle\theta_\eta = |\eta|e^{j\theta_\eta} \text{……… (22)}$$

With
$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \quad \tan 2\,\theta_\eta = \frac{\sigma}{\omega\varepsilon} \text{ ……… (23)}$$

Where $0 \leq \theta_\eta \leq 45^0$. Substituting equation (21) and (22) into equation (20) gives

$$\vec{H} = Re\left[\frac{E_0}{|\eta|e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)} \vec{a}_y\right]$$

Or $$\vec{H} = \frac{E_0}{|\eta|} e^{-\alpha z} cos\big(\omega t - \beta z - \theta_\eta\big)\vec{a}_y \ldots\ldots (24)$$

From equation (19) and (24), it is observed that the wave propagates along $\vec{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, and hence α is known as the **attenuation constant** or **attenuation coefficient** of the medium. It is measured in nepers per meter (Np/m) and can be expressed in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to e^{-1} of the original value, whereas an increase of 1 neper indicates an increase by a factor of e. Hence

$$1\ Np = 20\ \log_{10} e = 8.686\ dB \ldots.. (25)$$

From equation (13), we notice that if $\sigma = 0$, as is the case for a lossless medium and free space, $\alpha = 0$ and the wave is not attenuated as it propagates. The quantity β is a measure of the phase shift per unit length in radians per meter and is called the **phase constant** or **wave number**. In terms of β, the wave velocity u and wavelength λ are, given by

$$u = \frac{\omega}{\beta},\ \lambda = \frac{2\pi}{\beta}\ \ldots\ldots\ldots (26)$$

From equation (19) and (24), we notice that $\vec{E}\ and\ \vec{H}$ are out of phase by θ_η at any instant of time due to complex intrinsic impedance of the medium. Thus at any time, $\vec{E}\ leads\ \vec{H}$ (or$\vec{H}\ lags\ \vec{E}$) byθ_η.

We notice that the ratio of the magnitude of the conduction current density $\vec{J}_c$ to that of the displacement current density $\vec{J}_d$ in a lossy medium is

$$\frac{|\vec{J}_{cs}|}{|\vec{J}_{ds}|} = \frac{|\sigma\vec{E}_s|}{|j\omega\varepsilon\vec{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

Alternatively $$\tan\theta = \frac{\sigma}{\omega\varepsilon} \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

Where $\tan\theta$ is known as the **loss tangent** and θ is the **loss angle** of the medium. $\tan\theta$ or θ may be used to determine how lossy a medium is. A medium is said to be a good dielectric if $\tan\theta$ is very small ($\sigma \ll \omega\varepsilon$) or a good conductor if $\tan\theta$ is very large($\sigma \gg \omega\varepsilon$). From this we can say, a medium is regarded as a good conductor at low frequencies and may be a good dielectric at high frequencies. Note from equation (23) and (27) that

$$\theta = 2\theta_\eta \ldots.. (28)$$

Examples:

A lossy dielectric has an intrinsic impedance of $200 \angle 30^\circ\ \Omega$ at a particular frequency. If, at that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$\mathbf{H} = 10\, e^{-\alpha x} \cos\left(\omega t - \frac{1}{2} x\right) \mathbf{a}_y \text{ A/m}$$

find $\mathbf{E}$ and α. Determine the skin depth and wave polarization.

Solution:

The given wave travels along $\mathbf{a}_x$ so that $\mathbf{a}_k = \mathbf{a}_x$; $\mathbf{a}_H = \mathbf{a}_y$, so

$$-\mathbf{a}_E = \mathbf{a}_k \times \mathbf{a}_H = \mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$$

or

$$\mathbf{a}_E = -\mathbf{a}_z$$

Also $H_o = 10$, so

$$\frac{E_o}{H_o} = \eta = 200 \angle 30^\circ = 200\, e^{j\pi/6} \rightarrow E_o = 2000 e^{j\pi/6}$$

Except for the amplitude and phase difference, $\mathbf{E}$ and $\mathbf{H}$ always have the same form. Hence

$$\mathbf{E} = \mathrm{Re}\,(2000 e^{j\pi/6} e^{-\gamma x} e^{j\omega t} \mathbf{a}_E)$$

or

$$\mathbf{E} = -2 e^{-\alpha x} \cos\left(\omega t - \frac{x}{2} + \frac{\pi}{6}\right) \mathbf{a}_z \text{ kV/m}$$

Knowing that $\beta = 1/2$, we need to determine α. Since

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

and

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$$

$$\frac{\alpha}{\beta} = \left[\frac{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1}{\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1}\right]^{1/2}$$

But $\dfrac{\sigma}{\omega\varepsilon} = \tan 2\theta_\eta = \tan 60^\circ = \sqrt{3}$. Hence,

$$\frac{\alpha}{\beta} = \left[\frac{2 - 1}{2 + 1}\right]^{1/2} = \frac{1}{\sqrt{3}}$$

or

$$\alpha = \frac{\beta}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887 \text{ Np/m}$$

and

$$\delta = \frac{1}{\alpha} = 2\sqrt{3} = 3.464 \text{ m}$$

The wave has an E_z component; hence it is polarized along the z-direction.

PRACTICE EXERCISE 10.2

A plane wave propagating through a medium with $\varepsilon_r = 8$, $\mu_r = 2$ has $\mathbf{E} = 0.5\, e^{-z/3} \sin(10^8 t - \beta z)\, \mathbf{a}_x$ V/m. Determine

(a) β

(b) The loss tangent

(c) Wave impedance

(d) Wave velocity

(e) **H** field

Answer: (a) 1.374 rad/m, (b) 0.5154, (c) 177.72 $\angle 13.63°$ Ω, (d) 7.278×10^7 m/s, (e) $2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63°)\mathbf{a}_y$ mA/m.

P. E. 10.2 Let $x_o = \sqrt{1 + (\sigma / \omega\varepsilon)^2}$, then

$$\alpha = \omega \sqrt{\frac{\mu_o \varepsilon_o}{2} \mu_r \varepsilon_r (x_o - 1)} = \frac{\omega}{c} \sqrt{\frac{16}{2}} \sqrt{x_o - 1}$$

or $$\sqrt{x_o - 1} = \frac{\alpha c}{\omega \sqrt{8}} = \frac{1/3 \times 3 \times 10^8}{10^8 \sqrt{8}} = \frac{1}{\sqrt{8}} \longrightarrow x_o = 9/8$$

$$x_o^2 = \frac{81}{64} = 1 + (\sigma / \omega\varepsilon)^2 \longrightarrow \frac{\sigma}{\omega\varepsilon} = 0.5154$$

$$\tan 2\theta_\eta = 0.5154 \longrightarrow \theta_\eta = 13.63^o$$

$$\frac{\beta}{\alpha} = \sqrt{\frac{x_o + 1}{x_o - 1}} = \sqrt{17}$$

(a) $\beta = \alpha\sqrt{17} = \dfrac{\sqrt{17}}{3} = \underline{\underline{1.374 \text{ rad/m}}}$

(b) $\dfrac{\sigma}{\omega\varepsilon} = \underline{0.5154}$

(c) $|\eta| = \dfrac{\sqrt{\mu/\varepsilon}}{\sqrt{x_o}} = \dfrac{120\pi\sqrt{2/8}}{\sqrt{9/8}} = 177.72$

$$\eta = \underline{\underline{177.72 \angle 13.63° \,\Omega}}$$

(d) $u = \dfrac{\omega}{\beta} = \dfrac{10^8}{1.374} = \underline{\underline{7.278 \times 10^7}}$ m/s

(e) $\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E \longrightarrow \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_H \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_y$

$$\boldsymbol{H} = \frac{0.5}{177.5} e^{-z/3} \sin(10^8 t - \beta z - 13.63°)\boldsymbol{a}_y = \underline{2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63°)\boldsymbol{a}_y \text{ mA/m}}$$

4.6.2 Plane Waves in Lossless Dielectrics

In a lossless dielectric $\sigma \ll \omega\varepsilon$. It is a special case of lossy dielectric medium except that

$$\sigma \approx 0, \varepsilon = \varepsilon_0\varepsilon_r, \mu = \mu_0\mu_r \text{........ (29)}$$

Substituting these into equations (13) and (14) gives

$$\alpha = 0, \beta = \omega\sqrt{\mu\varepsilon} \text{ ..... (30)}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \lambda = \frac{2\pi}{\beta} \text{........ (31)}$$

Also
$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\angle 0^0 \text{ .... (32)}$$

And thus $\vec{E}$ *and* $\vec{H}$ are in time phase with each other.

4.6.3 Plane Waves in Free Space

Plane waves in free space comprise a special case of lossless dielectric medium except that

$$\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0 \text{........ (33)}$$

Substituting these into equations (13) and (14), we obtain

$$\alpha = 0,\ \beta = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c} \text{ ..... (34)}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c, \quad \lambda = \frac{2\pi}{\beta} \text{........ (35)}$$

Where $c = 3 \times 10^8$m/s, the speed of light in a vacuum. The fact that EM waves travel in free space at the speed of light.

By substituting the parameters in equation (33) into equation (23), $\theta_\eta = 0$ and $\eta = \eta_0$, where η_0 is called the **intrinsic impedance** of free space and is given by

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377\Omega \text{ .... (36)}$$

$$\vec{E} = E_0\cos(\omega t - \beta z)\vec{a}_x \text{...... (37)}$$

Then
$$\vec{H} = H_0\cos(\omega t - \beta z)\,\vec{a}_y = \frac{E_0}{\eta_0}\cos(\omega t - \beta z)\,\vec{a}_y \text{........ (38)}$$

In general, if $\vec{a}_E, \vec{a}_H$ *and* $\vec{a}_k$ are unit vectors along the $\vec{E}$ field, the $\vec{H}$ field, and the direction of wave propagation; it can be shown that

$$\vec{a}_k \times \vec{a}_E = \vec{a}_H$$

Or

$$\vec{a}_k \times \vec{a}_H = -\vec{a}_E$$

Or
$$\vec{a}_E \times \vec{a}_H = \vec{a}_k \text{ .... (39)}$$

Both $\vec{E}$ *and* $\vec{H}$ fields (or EM waves) are everywhere normal to the direction of wave propagation, $\vec{a}_k$. That means that the fields lie in a plane that is transverse or orthogonal to the direction of wave propagation. They form an EM wave that has no electric or magnetic field components along the direction of propagation; such a wave is called **transverse electromagnetic (TEM) wave**. A combination of $\vec{E}$ and $\vec{H}$ is called a **uniform plane wave** because $\vec{E}$ $(or\ \vec{H})$ has the same magnitude throughout any transverse plane, defined by $z = constant$. The direction in which the electric field points is the **polarization** of a TEM wave.

4.6.4 Plane Waves in Good Conductors

Plane waves in good conductors comprise another special case of lossy dielectric medium. A perfect, or good conductor, is one in which $\sigma \gg \omega\varepsilon$ so that $\frac{\sigma}{\omega\varepsilon} \gg 1$; that is

$$\sigma \approx \infty, \varepsilon = \varepsilon_0, \mu = \mu_0\mu_r \text{........ (40)}$$

Hence, equations (13) and (14) becomes

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \text{ ..... (41)}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta} \text{........ (42)}$$

Also, from equation (22),

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^0 \text{....... (43)}$$

And thus $\vec{E}$ $leads$ $\vec{H}$ by 45^0. If

$$\vec{E} = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\vec{a}_x \text{....... (44)}$$

Then
$$\vec{H} = \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} cos(\omega t - \beta z - 45^0)\vec{a}_y \text{...... (45)}$$

Therefore, as the $\vec{E}$ $(or\ \vec{H})$ wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ, through which the wave amplitude decreases to a factor e^{-1}(about 37% of the original value) is called **skin depth** or **penetration depth** of the medium; that is,

$$E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

Or
$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}} \text{ .... (46)}$$

The **skin depth** is a measure of the depth to which an EM wave can penetrate the medium. Noting that for good conductors we have $\alpha = \beta = \frac{1}{\delta}$, equation (44) can be written as

$$\vec{E} = E_0 e^{-\frac{z}{\delta}}\cos(\omega t - \frac{z}{\delta})\vec{a}_x$$

From equation (46), it is observed that the skin depth decreases with increasing frequency. The phenomenon whereby field intensity in a conductor rapidly decreases is known as **skin effect**. It is a tendency of charges to migrate from the bulk of the conducting material to the surface, resulting in higher resistance.

Applications of Skin Depth:

1. Because the skin depth in silver is very small, the difference in performance between a pure silver component and a silver-plated brass component is negligible, so silver plating is often used to reduce the material cost of waveguide components. For this same reason, hollow tubular conductors are used instead of solid conductors in outdoor television antennas.
2. The skin depth is useful in calculating the ac resistance due to skin effect. The dc resistance of a conductor wire of length 'l' and radius 'a' is given as $R_{dc} = \frac{l}{\sigma\pi a^2}$. When ac current flows in a conductor, it flows only on the skin of the conductor up-to the δ width instead of entire cross section. Hence $R_{ac} = \frac{l}{\sigma 2\pi a\delta}$.

So $$\frac{R_{ac}}{R_{dc}} = \frac{\frac{l}{\sigma 2\pi a\delta}}{\frac{l}{\sigma\pi a^2}} = \frac{a}{2\delta} = \frac{a}{2}\sqrt{\pi f\mu\sigma}$$

Example:1

In a lossless medium for which $\eta = 60\pi$, $\mu_r = 1$, and $\mathbf{H} = -0.1\cos(\omega t - z)\,\mathbf{a}_x + 0.5\sin(\omega t - z)\mathbf{a}_y$ A/m, calculate ε_r, ω, and $\mathbf{E}$.

Solution:

In this case, $\sigma = 0$, $\alpha = 0$, and $\beta = 1$, so

$$\eta = \sqrt{\mu/\varepsilon} = \sqrt{\frac{\mu_o}{\varepsilon_o}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}}$$

or

$$\sqrt{\varepsilon_r} = \frac{120\pi}{\eta} = \frac{120\pi}{60\pi} = 2 \qquad \rightarrow \qquad \varepsilon_r = 4$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{4} = \frac{2\omega}{c}$$

$$\omega = \frac{\beta c}{2} = \frac{1\,(3 \times 10^8)}{2} = 1.5 \times 10^8 \text{ rad/s}$$

From the given **H** field, **E** can be calculated in two ways: using the techniques (based on Maxwell's equations) developed in this chapter or directly using Maxwell's equations as in the last chapter.

Method 1: To use the techniques developed in this chapter, we let

$$\mathbf{E} = \mathbf{H}_1 + \mathbf{H}_2$$

where $\mathbf{H}_1 = -0.1\cos(\omega t - z)\,\mathbf{a}_x$ and $\mathbf{H}_2 = 0.5\sin(\omega t - z)\,\mathbf{a}_y$ and the corresponding electric field

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$$

where $\mathbf{E}_1 = E_{1o}\cos(\omega t - z)\,\mathbf{a}_{E_1}$ and $\mathbf{E}_2 = E_{2o}\sin(\omega t - z)\,\mathbf{a}_{E_2}$. Notice that although **H** has components along $\mathbf{a}_x$ and $\mathbf{a}_y$, it has no component along the direction of propagation; it is therefore a TEM wave.

For $\mathbf{E}_1$,

$$\mathbf{a}_{E_1} = -(\mathbf{a}_k \times \mathbf{a}_{H_1}) = -(\mathbf{a}_z \times -\mathbf{a}_x) = \mathbf{a}_y$$

$$E_{1o} = \eta\, H_{1o} = 60\pi\,(0.1) = 6\pi$$

Hence

$$\mathbf{E}_1 = 6\pi \cos(\omega t - z)\,\mathbf{a}_y$$

For $\mathbf{E}_2$,

$$\mathbf{a}_{E_2} = -(\mathbf{a}_k \times \mathbf{a}_{H_2}) = -(\mathbf{a}_z \times \mathbf{a}_y) = \mathbf{a}_x$$

$$E_{2o} = \eta H_{2o} = 60\pi\,(0.5) = 30\pi$$

Hence

$$\mathbf{E}_2 = 30\pi \sin(\omega t - z)\mathbf{a}_x$$

Adding $\mathbf{E}_1$ and $\mathbf{E}_2$ gives $\mathbf{E}$; that is,

$$\mathbf{E} = 94.25 \sin(1.5 \times 10^8 t - z)\,\mathbf{a}_x + 18.85 \cos(1.5 \times 10^8 t - z)\,\mathbf{a}_y \text{ V/m}$$

PRACTICE EXERCISE 10.3

A plane wave in a nonmagnetic medium has $\mathbf{E} = 50 \sin(10^8 t + 2z)\,\mathbf{a}_y$ V/m. Find

(a) The direction of wave propagation

(b) λ, f, and ε_r

(c) **H**

Answer: (a) along $-z$ direction, (b) 3.142 m, 15.92 MHz, 36, (c) $0.7958 \sin(10^8 t + 2z)\,\mathbf{a}_x$ A/m.

P. E. 10.3 (a) Along <u>-z direction</u>

(b) $\lambda = \dfrac{2\pi}{\beta} = 2\pi / 2 = \underline{\underline{3.142 \text{ m}}}$

$$f = \frac{\omega}{2\pi} = \frac{10^8}{2\pi} = \underline{\underline{15.92 \text{ MHz}}}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{(1)\varepsilon_r}$$

or $\sqrt{\varepsilon_r} = \beta c / \omega = \dfrac{3\times10^8\times2}{10^8} = 6 \longrightarrow \varepsilon_r = 36$

(c) $\theta_\eta = 0, |\eta| = \sqrt{\mu/\varepsilon} = \sqrt{\mu_o/\varepsilon_o}\sqrt{1/\varepsilon_r} = \dfrac{120\pi}{6} = 20\pi$

$\boldsymbol{a}_k = \boldsymbol{a}_E \times \boldsymbol{a}_H \longrightarrow -\boldsymbol{a}_z = \boldsymbol{a}_y \times \boldsymbol{a}_H \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_x$

$$\boldsymbol{H} = \frac{50}{20\pi}\sin(\omega t + \beta z)\boldsymbol{a}_x = \underline{\underline{795.8\sin(10^8 t + 2z)\boldsymbol{a}_x}} \text{ mA/m}$$

Example-3

A uniform plane wave propagating in a medium has

$$\mathbf{E} = 2e^{-\alpha z} \sin(10^8 t - \beta z)\, \mathbf{a}_y \text{ V/m}.$$

If the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$, and $\sigma = 3$ mhos/m, find α, β, and $\mathbf{H}$.

Solution:

We need to determine the loss tangent to be able to tell whether the medium is a lossy dielectric or a good conductor.

$$\frac{\sigma}{\omega\varepsilon} = \frac{3}{10^8 \times 1 \times \dfrac{10^{-9}}{36\pi}} = 3393 \gg 1$$

showing that the medium may be regarded as a good conductor at the frequency of operation. Hence,

$$\alpha = \beta = \sqrt{\frac{\mu\omega\sigma}{2}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)(3)}{2}\right]^{1/2}$$
$$= 61.4$$
$$\alpha = 61.4 \text{ Np/m}, \qquad \beta = 61.4 \text{ rad/m}$$

Also

$$|\eta| = \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi \times 10^{-7} \times 20(10^8)}{3}\right]^{1/2}$$
$$= \sqrt{\frac{800\pi}{3}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 3393 \quad \rightarrow \quad \theta_\eta = 45^\circ = \pi/4$$

Hence

$$\mathbf{H} = H_o e^{-\alpha z} \sin\left(\omega t - \beta z - \frac{\pi}{4}\right) \mathbf{a}_H$$

where

$$\mathbf{a}_H = \mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_z \times \mathbf{a}_y = -\mathbf{a}_x$$

and

$$H_o = \frac{E_o}{|\eta|} = 2\sqrt{\frac{3}{800\pi}} = 69.1 \times 10^{-3}$$

Thus

$$\mathbf{H} = -69.1\, e^{-61.4z} \sin\left(10^8 t - 61.42z - \frac{\pi}{4}\right) \mathbf{a}_x \text{ mA/m}$$

PRACTICE EXERCISE 10.4

A plane wave traveling in the $+y$-direction in a lossy medium ($\varepsilon_r = 4$, $\mu_r = 1$, $\sigma = 10^{-2}$ mhos/m) has $\mathbf{E} = 30\cos(10^9\pi\, t + \pi/4)\, \mathbf{a}_z$ V/m at $y = 0$. Find

(a) $\mathbf{E}$ at $y = 1$ m, $t = 2$ ns

(b) The distance traveled by the wave to have a phase shift of 10°

(c) The distance traveled by the wave to have its amplitude reduced by 40%

(d) $\mathbf{H}$ at $y = 2$ m, $t = 2$ ns

Answer: (a) $2.787\mathbf{a}_z$ V/m, (b) 8.325 mm, (c) 542 mm, (d) $-4.71\mathbf{a}_x$ mA/m

P. E. 10.4 (a)

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{10^9\pi \times 4 \times \dfrac{10^{-9}}{36\pi}} = 0.09$$

$$\alpha \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1 + \frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 - 1\right]} = \frac{\omega}{2c}\sqrt{\mu_r\varepsilon_r}\frac{\sigma}{\omega\varepsilon} = \frac{10^9\pi}{2\times3\times10^8}(2)(0.09) = 0.9425 \text{ Np/m}$$

$$\beta \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1 + \frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + 1\right]} = \frac{10^9\pi}{3\times10^8}\sqrt{2[2 + 0.5(0.09)^2]} = 20.965 \text{ rad/m}$$

$$\boldsymbol{E} = 30e^{-0.9425y}\cos(10^9\pi t - 20.96y + \pi/4)\boldsymbol{a}_z$$

At t = 2ns, y = 1m,

$$\boldsymbol{E} = 30e^{-0.9425}\cos(2\pi - 20.96 + \pi/4)\boldsymbol{a}_z = \underline{\underline{2.844\boldsymbol{a}_z}} \quad \text{V/m}$$

(b) $\beta y = 10^o = \dfrac{10\pi}{180}$ rad

or

$$y = \frac{\pi}{18}\frac{1}{\beta} = \frac{\pi}{18\times20.965} = \underline{8.325 \text{ mm}}$$

(c) $30(0.6) = 30\ e^{-\alpha y}$

$$y = \frac{1}{\alpha}\ln(1/0.6) = \frac{1}{0.9425}\ln\frac{1}{0.6} = \underline{542 \text{ mm}}$$

(d)

$$|\eta| \cong \frac{\sqrt{\mu/\varepsilon}}{[1 + \frac{1}{4}(0.09)^2]} = \frac{60\pi}{1.002} = 188.11\ \Omega$$

$$2\theta_\eta = \tan^{-1}0.09 \longrightarrow \theta_\eta = 2.571^o$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$$

$$\boldsymbol{H} = \frac{30}{188.11}e^{-0.9425y}\cos(10^9\pi t - 20.96y + \pi/4 - 2.571^o)\boldsymbol{a}_x$$

At y = 2m, t = 2ns,

$$\boldsymbol{H} = (0.1595)(0.1518)\ \cos(-34.8963rad)\boldsymbol{a}_x = \underline{\underline{-22.83\boldsymbol{a}_x}} \text{ mA/m}$$

4.6.5 Power and the Poynting Vector

As we know, energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla \times \vec{E} = -\mu\frac{\partial \vec{H}}{\partial t} \ldots\ldots\ldots (1)$$

$$\nabla \times \vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial \vec{E}}{\partial t} \ldots\ldots\ldots (2)$$

Dotting both sides of equation (2) with $\vec{E}$ gives

$$\vec{E}.\left(\nabla\times\vec{H}\right)=\sigma E^2+\vec{E}.\varepsilon\frac{\partial\vec{E}}{\partial t}\ldots\ldots\ldots (3)$$

But for any vector fields $\vec{A}$ and $\vec{B}$

$$\nabla.\left(\vec{A}\times\vec{B}\right)=\vec{B}.\left(\nabla\times\vec{A}\right)-\vec{A}.\left(\nabla\times\vec{B}\right)$$

Applying this vector identity to equation (3) ($letting \vec{A}=\vec{H}\ and\ \vec{B}=\vec{E}$) gives

$$\vec{H}.\left(\nabla\times\vec{E}\right)+\nabla.\left(\vec{H}\times\vec{E}\right)=\vec{E}.\left(\nabla\times\vec{H}\right)=\ \sigma E^2+\vec{E}.\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$=\sigma E^2+\frac{1}{2}.\varepsilon\frac{\partial E^2}{\partial t}\ \ldots\ldots. (4)$$

Since $\frac{\partial E^2}{\partial t}=\frac{\partial}{\partial t}\left(\vec{E}.\vec{E}\right)=\vec{E}.\frac{\partial\vec{E}}{\partial t}+\vec{E}.\frac{\partial\vec{E}}{\partial t}=2\vec{E}.\frac{\partial\vec{E}}{\partial t}$

Dotting both sides of equation (1) with $\vec{H}$, we write

$$\vec{H}.\left(\nabla\times\vec{E}\right)=\vec{H}.\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)=-\frac{\mu}{2}\frac{\partial}{\partial t}(\vec{H}.\vec{H})\ldots\ldots\ldots. (5)$$

And thus equation (4) becomes

$$-\frac{\mu}{2}\frac{\partial H^2}{\partial t}-\nabla.\left(\vec{E}\times\vec{H}\right)=\ \sigma E^2+\frac{1}{2}.\varepsilon\frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the volume integral of both sides,

$$\int_v\nabla.\left(\vec{E}\times\vec{H}\right)dv=-\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right]dv-\int_v\sigma E^2dv\ldots\ldots. (6)$$

Applying the divergence theorem to the left hand side gives

$$\oint_S\left(\vec{E}\times\vec{H}\right).d\vec{S}=-\frac{\partial}{\partial t}\left[\frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2\right]dv-\int_v\sigma E^2dv\ \ldots\ldots. (7)$$

↓ ↓ ↓

Total power leaving the volume	=	rate of decrease in energy stored in electric and magnetic fields	-	ohmic power dissipated	(8)

Equation (7) is referred to as **Poynting's theorem**. The quantity $\vec{E}\times\vec{H}$ on the left hand side of equation (7) is known as **Poynting vector** $\mathscr{P}$, measured in watts per square meter (W/m^2); that is

$$\mathscr{P}=\left(\vec{E}\times\vec{H}\right)\ldots\ldots\ldots. (9)$$

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the ohmic losses.

It should be noted that $\mathscr{P}$ is normal to both $\vec{E}\ and\ \vec{H}$ and is therefore along the direction of wave propagation $\vec{a}_K$ for uniform plane waves. Thus

$$\vec{a}_K=\vec{a}_E\times\vec{a}_H\ldots\ldots.. (10)$$

The fact that $\mathscr{P}$ points along $\vec{a}_K$ causes $\mathscr{P}$ to be regarded as a "pointing" vector.

If we assume that

$$\vec{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \vec{a}_x$$

Then

$$\vec{H}(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \vec{a}_y$$

And

$$\mathscr{P}(z,t) = \frac{{E_0}^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta)\, \vec{a}_z$$

$$= \frac{{E_0}^2}{|\eta|} e^{-2\alpha z} [\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)] \vec{a}_z \text{ .... (11)}$$

Since $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$.

To determine the time average Poynting vector $\mathscr{P}_{ave}\ (z,t)$ in (W/m^2), which is obtained by integrating equation (11) over the period $T = 2\pi/\omega$; that is,

$$\mathscr{P}_{ave}\ (z,t) = \frac{1}{T}\int_0^T \mathscr{P}(z,t)\, dt \text{........ (12)}$$

By substituting eq. (11) into eq. (12), we obtain

$$\mathscr{P}_{ave}\ (z) = \frac{{E_0}^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta\, \vec{a}_z \text{...... (13)}$$

The total time-average power crossing a given surface S is given by

$$P_{ave} = \oint_S \mathscr{P}_{ave} \,.\, \mathrm{dS}$$

Example 10

In a nonmagnetic medium

$$\mathbf{E} = 4 \sin(2\pi \times 10^7 t - 0.8x)\, \mathbf{a}_z \text{ V/m}$$

Find

(a) ε_r, η

(b) The time-average power carried by the wave

(c) The total power crossing 100 cm^2 of plane $2x + y = 5$

Solution:

(a) Since $\alpha = 0$ and $\beta \neq \omega/c$, the medium is not free space but a lossless medium.

$$\beta = 0.8, \quad \omega = 2\pi \times 10^7, \quad \mu = \mu_o \text{ (nonmagnetic)}, \quad \varepsilon = \varepsilon_o\varepsilon_r$$

Hence

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

or

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{0.8(3 \times 10^8)}{2\pi \times 10^7} = \frac{12}{\pi}$$

$$\varepsilon_r = 14.59$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_o}{\varepsilon_o\varepsilon_r}} = \frac{120\pi}{\sqrt{\varepsilon_r}} = 120\pi \cdot \frac{\pi}{12} = 10\pi^2$$

$$= 98.7\ \Omega$$

(b) $\mathcal{P} = \mathbf{E} \times \mathbf{H} = \dfrac{E_o^2}{\eta} \sin^2(\omega t - \beta x)\, \mathbf{a}_x$

$$\mathcal{P}_{\text{ave}} = \frac{1}{T}\int_0^T \mathcal{P}\, dt = \frac{E_o^2}{2\eta}\,\mathbf{a}_x = \frac{16}{2 \times 10\pi^2}\,\mathbf{a}_x$$
$$= 81\ \mathbf{a}_x\ \text{mW/m}^2$$

(c) On plane $2x + y = 5$ (see Example 3.5 or 8.5),

$$\mathbf{a}_n = \frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}$$

Hence the total power is

$$P_{\text{ave}} = \int \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} = \mathcal{P}_{\text{ave}} \cdot S\,\mathbf{a}_n$$
$$= (81 \times 10^{-3}\mathbf{a}_x) \cdot (100 \times 10^{-4}) \left[\frac{2\mathbf{a}_x + \mathbf{a}_y}{\sqrt{5}}\right]$$
$$= \frac{162 \times 10^{-5}}{\sqrt{5}} = 724.5\ \mu\text{W}$$