ELECTRO MAGNETIC FIELDS
(3-0-0)
LECTURE NOTES
B. TECH
(II YEAR – III SEM)

Prepared by:

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Electromagnetic Fields (3-0-0)

Prerequisites:

1. Mathematics-I
2. Mathematics-II

Course Outcomes

At the end of the course, students will demonstrate the ability

1. To understand the basic laws of electromagnetism.
2. To obtain the electric and magnetic fields for simple configurations under static conditions.
3. To analyse time-varying electric and magnetic fields.
4. To understand Maxwell's equation in different forms and different media.
5. To understand the propagation of EM waves.

Module 1: (08 Hours)

Co-ordinate systems & Transformation: Cartesian co-ordinates, circular cylindrical coordinates, spherical coordinates. Vector Calculus: Differential length, Area & Volume, Line, surface and volume Integrals, Del operator, Gradient of a scalar, Divergence of a vector & Divergence theorem, Curl of a vector & Stoke's theorem, Laplacian of a scalar.

Module 2: (10 Hours)

Electrostatic Fields: Coulomb's Law, Electric Field Intensity, Electric Fields due to a point, line, surface and volume charge, Electric Flux Density, Gauss's Law- Maxwell's Equation, Applications of Gauss's Law, Electric Potential, Relationship between E and V- Maxwell's Equation and Electric Dipole & Flux Lines, Energy Density in Electrostatic Fields., Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions. Electrostatic boundary-value problems: Poisson's and Laplace's Equations, Uniqueness Theorem, General procedures for solving Poisson's and Laplace's equations, Capacitance.

Module 3: (06 Hours)

Magneto static Fields: Magnetic Field Intensity, Biot-Savart's Law, Ampere's circuit Law-Maxwell Equation, applications of Ampere's law, Magnetic Flux Density-Maxwell's equations. Maxwell's equation for static fields, Magnetic Scalar and Vector potentials. Magnetic Boundary Conditions.

Module 4: (10 Hours)

Electromagnetic Field and Wave propagation: Faraday's Law, Transformer & Motional Electromagnetic Forces, Displacement Current, Maxwell's Equation in Final forms, Time-Harmonic Field. Electromagnetic Wave Propagation: Wave Propagation in lossy Dielectrics, Plane Waves in loss less Dielectrics, Free space, Good conductors Power & Poynting vector.

TEXTBOOKS:

1. Matthew N. O. Sadiku, Principles of Electromagnetics, 6th Ed., Oxford Intl. Student Edition, 2014.

REFERENCE BOOKS:

1. C. R. Paul, K. W. Whites, S. A. Nasor, Introduction to Electromagnetic Fields, 3rd Ed, TMH.
2. W.H. Hyat, Electromagnetic Field Theory, 7th Ed, TMH.
3. A. Pramanik, "Electromagnetism - Theory and applications", PHI Learning Pvt. Ltd, New Delhi, 2009.
4. A. Pramanik, "Electromagnetism-Problems with solution", Prentice Hall India, 2012.
5. G.W. Carter, "The electromagnetic field in its engineering aspects", Longmans, 1954.
6. W.J. Duffin, "Electricity and Magnetism", McGraw Hill Publication, 1980.
7. W.J. Duffin, "Advanced Electricity and Magnetism", McGraw Hill, 1968.
8. E.G. Cullwick, "The Fundamentals of Electromagnetism", Cambridge University Press, 1966.
9. B. D. Popovic, "Introductory Engineering Electromagnetics", Addison- Wesley Educational Publishers, International Edition, 1971.
10. W. Hayt, "Engineering Electromagnetics", McGraw Hill Education, 2012.

MODULE-III

1. Magneto static Fields
    - Magnetic Field Intensity
    - Biot-Savart's Law
    - Ampere's Circuit Law-Maxwell's Equations
    - Maxwell's Equation for Static Field
    - Magnetic Scalar and Vector Potentials
    - Magnetic Boundary Conditions

3.1 Introduction

The Scientist Oersted stated that when the charges are in motion, they are surrounded by a magnetic field. The charges in motion i.e. flow of charges constitutes an electric current. Thus a current carrying conductor is always surrounded by a magnetic field. If such a current flow is steady i.e. time invariant, then the magnetic field produced is a steady magnetic field which is also a time invariant. The direct current (d.c.) is a steady flow of current hence magnetic field produced by a conductor carrying a d.c. current is a static steady magnetic field. The study of steady magnetic field, existing in a given space, produced due to the flow of direct current through a conductor is called **magnetostatics**.

3.1.1Magnetic Field and its Properties

The region around a magnet within which the influence of the magnet can be experienced is called **magnetic field**. Such a field is represented by imaginary lines around the magnet which are called **magnetic lines of force**. The direction of such lines is always from N pole to S pole, external to the magnet as shown in the Fig. 3.1.1. These lines of force are also called **magnetic lines of flux** or **magnetic flux lines**.

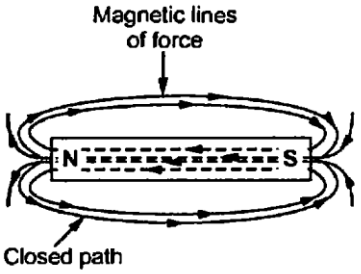


Fig.3.1.1 Permanent magnet and magnetic lines of force

An important difference between electric flux lines and magnetic flux lines can be observed here. ln case of electric flux, the flux lines originate from an isolated positive charge and diverge to terminate at infinity. While for a negative charge, electric flux lines converge on a charge, starting from infinity. But in case of magnetic flux, the poles exist in pairs only. Hence every magnetic flux line starting from north pole must end at south pole and complete the path from **south to north internal to the magnet**.

Key Point: An isolated magnetic pole cannot exist. The magnetic flux lines exist in the form of closed loop.

3.1.2 Magnetic Field Intensity

The quantitative measure of strongness or weakness of the magnetic field is given by magnetic field intensity or magnetic field strength. The **magnetic field intensity** at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength when placed at that point. The magnetic flux lines are measured in webers (Wb) while magnetic field intensity is measured in newtons/weber (N/Wb) or amperes per metre (A/m) or ampere-turns/metre (AT/m). It is denoted as $\vec{H}$. It is a vector quantity. This is similar to the electric field intensity $\vec{E}$ in electrostatics.

3.1.3 Magnetic Flux Density

The total magnetic lines of force i.e. magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called **magnetic flux density**. it is denoted as $\vec{B}$ and is a vector quantity. It is measured in weber per square metre. (Wb/m^2) which is also called Tesla (T). This is similar to the electric flux density $\vec{D}$ in electrostatics.

3.1.4 Relation between $\vec{B}$ and $\vec{H}$

ln electrostatics, $\vec{E}$ and $\vec{D}$ are related to each other through permittivity ε of the region. In magnetostatics, the $\vec{B}$ and $\vec{H}$ are related to each other through the property of the region in which current carrying conductor is placed. It is called **permeability** denoted as μ. It is the ability or ease with which the current carrying conductor forces the magnetic flux through the region around it. For a free space, the permeability is denoted as μ_0 and its value is $4\pi \times 10^{-7}$. As ε is measured in F/m, the perrneability it is measured in henries per metre (H/m). For any other region, a relative permeability is specified as μ_r and $\mu = \mu_0\mu_r$.

The $\vec{B}$ and $\vec{H}$ are related as,

$$\vec{B} = \mu\vec{H} = \mu_0\mu_r\vec{H}$$

For free space, $\vec{B} = \mu_0\vec{H}$ and for all nonmagnetic media, $\mu_r = 1$ while for magnetic materials μ_r is greater than unity.

3.2 Biot-Savart Law

Consider a conductor carrying a direct current I and a steady magnetic field produced around it. The Biot-Savart law allows us to obtain the **differential magnetic field intensity** $d\vec{H}$, produced at a point P, due to a differential current element $Id\vec{L}$. The current carrying conductor is shown in the Fig.3.2.1.

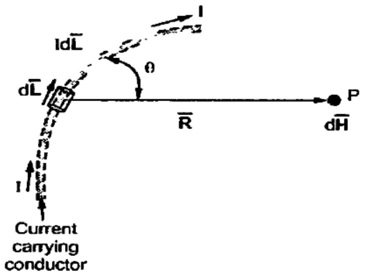


Fig.3.2.1

Consider a differential length dL, hence the differential current clement is IdL. This is very small part of the current carrying conductor. The point P is at a distance R from the differential current element. The θ is the angle between the differential current element and the line joining point P to the differential current element.

The Biot-Savart law states that, the magnetic field intensity $d\vec{H}$ produced at a point P due to a differential current element IdL is,

1. Proportional to the product of the current I and differential length dL.
2. The sine of the angle between the element and the line joining point P to the element.
3. And inversely proportional to the square of the distance R between point P and the element.

Mathematically, the Biot-Savart law can be stated as,

$$dH \propto \frac{IdL\ sin\theta}{R^2}$$

$$d\vec{H} = \frac{kIdL\ sin\theta}{R^2}$$

Where $k =$ constant of proportionality

In SI units, $k = \frac{1}{4\pi}$

$\therefore$ $dH = \frac{IdL\ sin\theta}{4\pi R^2}$........... (1)

Let us express this equation in vector form.

Let dL = Magnitude of vector length $d\vec{L}$ and

$\vec{a}_R$ = Unit vector in the direction from differential current element to point P

Then from rule of cross product,

$$d\vec{L} \times \vec{a}_R = dL|\vec{a}_R|sin\theta = dLsin\theta \qquad \text{since } |\vec{a}_R| = 1$$

Replacing in equation (1),

$$d\vec{H} = \frac{Id\vec{L}\times\vec{a}_R}{4\pi R^2} \text{ A/m........... (2)}$$

But $\vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$

Hence $d\vec{H} = \frac{Id\vec{L}\times\vec{R}}{4\pi R^3}$ A/m........... (3)

The equations (2) and (3) is the mathematical form of Biot-Savart law.

According to the direction of cross product, the direction of $d\vec{H}$ is normal to the plane containing two vectors and in that normal direction which is along the progress of right handed screw, turned from $d\vec{L}$ through the smaller angle θ towards the line joining element to the point P. Thus the direction of $d\vec{H}$ is normal to the plane of paper. For the case considered, according to right handed screw rule, the direction of $d\vec{H}$ is going into the plane of the paper.

The entire conductor is made up of all such differential elements. Hence to obtain total magnetic field intensity $\vec{H}$ the above equation (2) takes the integral form as,

$$\vec{H} = \int \frac{Id\vec{L}\times\vec{a}_R}{4\pi R^2} \text{ A/m........... (4)}$$

This is called **Integral form of Biot-Savart law.**

3.2.1 Biot-Savart Law ln-terms of Distributed Sources

Consider a surface carrying a uniform current over its surface as shown in the Fig.3.2.2. Then the surface current density is denoted as $\vec{K}$ and measured in amperes per metre (A/m). Thus for uniform current I in any width b is given by $I = Kb$ Where width b is perpendicular to the direction of current flow.

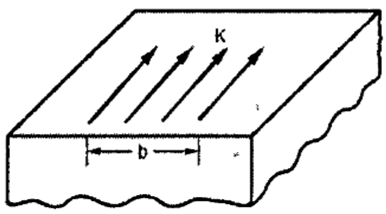


Fig.3.2.2 Surface current density

Thus if dS is the differential surface area considered of a surface having current density $\vec{K}$ then,

$$Id\vec{L} = \vec{K}dS \ldots\ldots (5)$$

If the current density in a volume of a given conductor is $\vec{J}$ measured in A / m^2 then for a differential volume dv we can write,

$$Id\vec{L} = \vec{J}dv \ldots\ldots (6)$$

Hence the Biot-Savart law can be expressed for surface current considering $\vec{K}dS$ while for volume current considering $\vec{J}dv$.

$\therefore$
$$\vec{H} = \int_S \frac{\vec{K}\times\vec{a}_r\, dS}{4\pi R^2} \text{ A/m} \ldots\ldots\ldots\ldots (7)$$

and
$$\vec{H} = \int_v \frac{\vec{J}\times\vec{a}_r\, dv}{4\pi R^2} \text{ A/m} \ldots\ldots\ldots\ldots (8)$$

3.2.2 H due to Straight Conductor of Finite Length

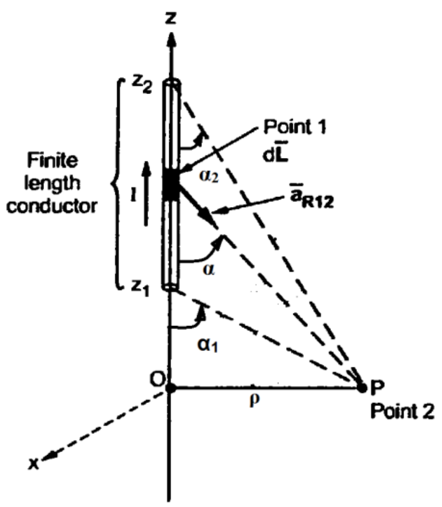


Fig. 3.2.3

Consider a conductor of finite length placed along z-axis, as shown in the Fig. 3.2.3. It carries a direct current I. The perpendicular distance of point P from z-axis is ρ. The conductor is placed such that its one end is at $z = z_1$ while other at $z = z_2$.

Consider a differential element $d\vec{L}$ along z-axis, at a distance z from origin.

$\therefore$
$$d\vec{L} = dz\vec{a}_z \ldots\ldots (1)$$

The unit vector in the direction joining differential element to point P is $\vec{a}_{R12}$ and can be expressed as

$$\vec{a}_{R12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\rho\vec{a}_\rho - z\vec{a}_z}{\sqrt{(\rho)^2+(-z)^2}} = \frac{\rho\vec{a}_\rho - z\vec{a}_z}{\sqrt{\rho^2+z^2}} \ldots.. (2)$$

$\therefore$
$$d\vec{L} \times \vec{a}_{R12} = \frac{\rho dz}{\sqrt{\rho^2+z^2}}\vec{a}_\phi$$

According to Biot-Savart law, $d\vec{H}$ at point P is,

$$d\vec{H} = \frac{Id\vec{L} \times \vec{a}_{R12}}{4\pi {R_{12}}^2}$$

$$= \frac{I\rho dz\vec{a}_\phi}{4\pi\sqrt{\rho^2+z^2}(\sqrt{\rho^2+z^2})^2}$$

$$= \frac{I\rho dz\vec{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}} \ldots\ldots\ldots.. (3)$$

The total $\vec{H}$ at P due to conductor of finite length can be obtained by integrating $d\vec{H}$ over $z = z_1$ to $z = z_2$.

$$\vec{H} = \int_{z_1}^{z_2} d\vec{H} = \int_{z_1}^{z_2} \frac{I\rho dz \vec{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}} \ldots\ldots (4)$$

Let $z = \rho cot\alpha$ hence $dz = -\rho cosec^2\alpha d\alpha$ and $(\rho^2 + z^2)^{3/2} = \rho^3 cosec^3\alpha$ then equation (4) becomes

$$\vec{H} = \int_{z_1}^{z_2} d\vec{H} = -\frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2 cosec^2\alpha\, d\alpha}{\rho^3 cosec^3\alpha}\vec{a}_\phi$$

$$= -\frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\rho cosec\alpha}\vec{a}_\phi = -\frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} sin\alpha d\alpha\, \vec{a}_\phi$$

$$\therefore \quad \vec{H} = \frac{I}{4\pi\rho}(cos\alpha_2 - cos\alpha_1)\vec{a}_\phi \ldots\ldots\ldots\ldots\ldots (5)$$

$$\therefore \quad \vec{B} = \frac{\mu I}{4\pi\rho}(cos\alpha_2 - cos\alpha_1)\vec{a}_\phi \ldots\ldots\ldots\ldots\ldots (6)$$

Key Points: Equation (5) and (6) is applicable to straight filamentary conductor of finite length. The conductor need not lie on z-axis but it must be straight. The direction of $\vec{H}$ and $\vec{B}$ are always $\vec{a}_\phi$ i.e. along concentric circular path irrespective of length of wire or point of interest.

Special cases:

1. When the conductor is semi-infinite with respect to P so that point z_1 is now at (0, 0, 0) while z_2 is at (0, 0, ∞), $\alpha_1 = 90^0$, $\alpha_2 = 0^0$ and equation (5) and (6) become

$$\therefore \quad \vec{H} = \frac{I}{4\pi\rho}\vec{a}_\phi \ldots\ldots\ldots\ldots\ldots (7)$$

$$\therefore \quad \vec{B} = \frac{\mu I}{4\pi\rho}\vec{a}_\phi \ldots\ldots\ldots\ldots\ldots (8)$$

2. When the conductor is of infinite with respect to P so that point z_1 is now at (0, 0, -∞) while z_2 is at (0, 0, ∞), $\alpha_1 = 180^0$, $\alpha_2 = 0^0$ and equation (5) and (6) become

$$\therefore \quad \vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi \ldots\ldots\ldots\ldots\ldots (9)$$

$$\therefore \quad \vec{B} = \frac{\mu I}{2\pi\rho}\vec{a}_\phi \ldots\ldots\ldots\ldots\ldots (10)$$

Hence $\vec{a}_\phi$ can be determine as , $\vec{a}_\phi = \vec{a}_l \times \vec{a}_\rho$…….. (11)

Where $\vec{a}_l$ is a unit vector along the line current and $\vec{a}_\rho$ is a unit vector along the perpendicular line from the line current to field point.

3.2.3 $\vec{\mathbf{H}}$ on the Axis of a Circular Loop

Consider a circular loop carrying a direct current I, placed in xy plane, with z-axis as its axis as shown in the Fig. 3.2.4. The magnetic field intensity $\vec{H}$ at point P is to be obtained. The point P is at a distance z from the plane of the circular loop, along it’s axis.

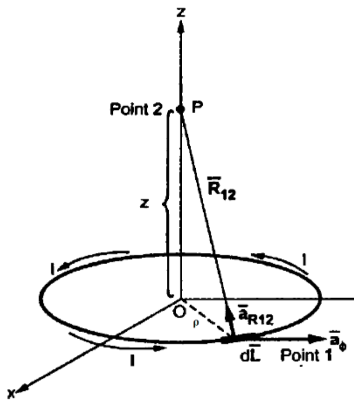


Fig. 3.2.4

The radius of the circular loop is 'ρ'. Consider the differential length $d\vec{L}$ of the circular loop as shown in the Fig. 3.2.4.

In the cylindrical co-ordinate system,

$$d\vec{L} = d\rho\vec{a}_\rho + \rho d\phi\vec{a}_\phi + dz\vec{a}_z$$

But loop is in the plane for which ρ is constant and $z = o =$ constant plane. The I$d\vec{L}$ is tangential at point 1 in $\vec{a}_\phi$ direction.

$$\therefore \quad Id\vec{L} = I\rho d\phi\vec{a}_\phi \text{...... (12)}$$

The unit vector $\vec{a}_{R12}$ is in the direction along the line joining differential current element to the point P.

$$\therefore \quad \vec{a}_{R12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|}$$

From the Fig.3.2.4, it can be observed that,

$$\vec{R}_{12} = -\rho\vec{a}_\rho + z\vec{a}_z \quad \text{From point 1 to point 2}$$

Hence

$$|\vec{R}_{12}| = \sqrt{(-\rho)^2 + z^2} = \sqrt{\rho^2 + z^2}$$

$$\therefore \quad \vec{a}_{R12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{-\rho\vec{a}_\rho + z\vec{a}_z}{\sqrt{\rho^2+z^2}} \text{........ (13)}$$

$$\text{Now} \quad d\vec{L} \times \vec{a}_{R12} = \begin{vmatrix} \vec{a}_\rho & \vec{a}_\phi & \vec{a}_z \\ d\rho & \rho d\phi & dz \\ -\rho & 0 & z \end{vmatrix} = z\rho d\phi\vec{a}_\rho + \rho^2 d\phi\vec{a}_z$$

Note that while calculating cross product $|\vec{R}_{12}|$ is neglected for convenience, which must be considered in further calculations.

According to Biot-Savart law, the differential field strength $d\vec{H}$ at point P is given by,

$$d\vec{H} = \frac{Id\vec{L}\times\vec{a}_{R12}}{4\pi {R_{12}}^2} = \frac{I(z\rho d\phi\vec{a}_\rho + \rho^2 d\phi\vec{a}_z)}{4\pi\sqrt{\rho^2+z^2}\left(\sqrt{\rho^2+z^2}\right)^2} \text{...... (14)}$$

The total $\vec{H}$ is to be obtained by integrating $d\vec{H}$ over the circular loop i.e. for $\phi = 0$ to 2π.

Note : It can be observed that though $d\vec{H}$ consists of two components $\vec{a}_\rho$and $\vec{a}_z$, due to radial symmetry all $\vec{a}_\rho$ components are going to cancel each other. So $\vec{H}$ exists only along the axis in it direction.

$$\therefore \quad \vec{H} = \frac{I}{4\pi}\int_{\phi=0}^{\phi=2\pi} \frac{\rho^2 d\phi\vec{a}_z}{(\rho^2+z^2)^{3/2}}$$

$$= \frac{I\rho^2\vec{a}_z}{4\pi(\rho^2+z^2)^{3/2}}\int_0^{2\pi} d\phi$$

$$= \frac{I\rho^2 \vec{a}_z \times 2\pi}{4\pi(\rho^2 + z^2)^{3/2}}$$

$\therefore$ $$\vec{H} = \frac{I\rho^2}{2(\rho^2+z^2)^{3/2}} \vec{a}_z \text{ A/m} \ldots\ldots\ldots (15)$$

where r = Radius of the circular loop

z = Distance of point P along the axis

Note: If point P is shifted at the centre of the circular loop i.e. z = 0, we get the result obtained in earlier section.

$$\vec{H} = \frac{I}{2\rho} \vec{a}_z \text{ A/m} \ldots\ldots\ldots (16)$$

where $\vec{a}_z$ is the unit vector normal to xy plane in which the circular loop is lying.

Example 7.2 : *A current filament carries a current of 10 A in the $\bar{a}_z$ direction on the z-axis. Find the magnetic field intensity $\bar{H}$ at point P (1,2,3) due to this filament if it extends from,*

a) $z = -\infty$ to ∞ b) $z = 0$ to 5 m c) $z = 5$ to ∞.

Express answers in cartesian co-ordinates.

Solution : The arrangements are shown in the Fig. 7.18.

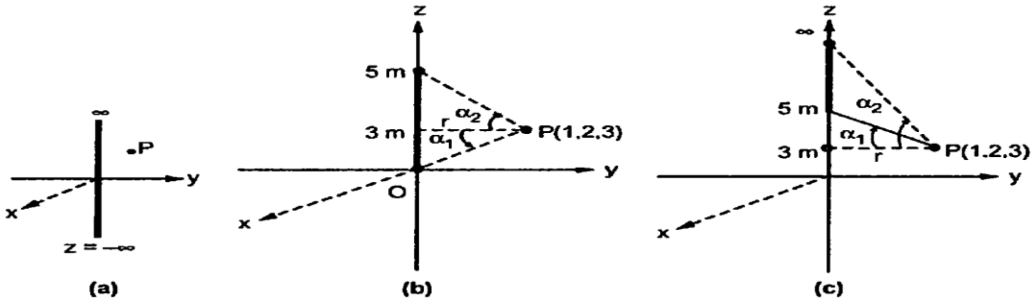


Fig. 7.18

Case a : It is infinitely long straight conductor.

$$\bar{H} = \frac{I}{2\pi r} \bar{a}_\phi , \qquad P\ (1, 2, 3), \quad I = 10 \text{ A}$$

Now $$r = \sqrt{x^2+y^2} = \sqrt{1+4} = \sqrt{5} \text{ m}$$

$\therefore$ $$\bar{H} = \frac{10}{2\pi \times \sqrt{5}} \bar{a}_\phi = 0.7117\ \bar{a}_\phi \text{ A/m}$$

To find x component, take dot product with $\bar{a}_x$.

$\therefore$ $$H_x = \bar{H} \cdot \bar{a}_x = 0.7117\ \bar{a}_\phi \cdot \bar{a}_x = -0.7117 \sin\phi$$

Similarly $$H_y = \bar{H} \cdot \bar{a}_y = 0.7117\ \bar{a}_\phi \cdot \bar{a}_y = +0.7117 \cos\phi$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.43^\circ \quad \text{... For point P}$$

$\therefore$ $$H_x = -0.6365, \quad H_y = 0.3183$$

$\therefore$ $$\bar{H} = -0.6365\ \bar{a}_x + 0.3183\ \bar{a}_y \text{ A/m}$$

Case b : It is a finite length conductor with $z_1 = 0$ and $z_2 = 5$ m.

$$r = \sqrt{x^2+y^2} = \sqrt{1+4} = \sqrt{5} \text{ m}$$

$$\alpha_1 = \tan^{-1}\frac{3}{\sqrt{5}} = 53.3^\circ$$

but negative as that end is below point P.

$\therefore$ $$\alpha_1 = -53.3^\circ$$

$$\alpha_2 = \tan^{-1}\frac{2}{\sqrt{5}} = 41.81^\circ$$

Now $\quad \bar{H} = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1]\,\bar{a}_\phi$

$$= \frac{10}{4\pi\times\sqrt{5}}[\sin 41.81° - \sin(-53.3°)]\bar{a}_\phi = 0.5225\,\bar{a}_\phi$$

$\therefore \quad H_x = \bar{H}\cdot\bar{a}_x = 0.5225\,(\bar{a}_\phi\cdot\bar{a}_x) = 0.5225\,(-\sin\phi)$

and $\quad H_y = \bar{H}\cdot\bar{a}_y = 0.5225\,(\bar{a}_\phi\cdot\bar{a}_y) = 0.5225\,(\cos\phi)$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.43° \qquad \text{... For point P}$$

$\therefore \quad H_x = -0.4673,\quad H_y = 0.2337$

$\therefore \quad \bar{H} = \mathbf{-0.4673\,\bar{a}_x + 0.2337\,\bar{a}_y\ \ A/m}$

Case c : It is a conductor from z = 5 to z = ∞.

$$r = \sqrt{x^2+y^2} = \sqrt{1+4} = \sqrt{5}\ \text{m}$$

$$\alpha_1 = \tan^{-1}\frac{2}{r} = \tan^{-1}\frac{2}{\sqrt{5}} = 41.81°$$

$$\alpha_2 = \tan^{-1}\frac{\infty}{r} = 90°$$

Both α_1 and α_2 are positive as above point P.

$\therefore \quad \bar{H} = \frac{I}{4\pi r}[\sin\alpha_2 - \sin\alpha_1]\bar{a}_\phi$

$$= \frac{10}{4\pi\times\sqrt{5}}[\sin 90 - \sin 41.81]\bar{a}_\phi = 0.1186\,\bar{a}_\phi$$

$\therefore \quad H_x = \bar{H}\cdot\bar{a}_x = 0.1186\,(\bar{a}_\phi\cdot\bar{a}_x) = 0.1186\,(-\sin\phi)$

and $\quad H_y = \bar{H}\cdot\bar{a}_y = 0.1186\,(\bar{a}_\phi\cdot\bar{a}_y) = 0.1186\,(\cos\phi)$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{1} = 63.43° \qquad \text{... For point P}$$

$\therefore \quad H_x = -0.106,\quad H_y = 0.053$

$\therefore \quad \bar{H} = \mathbf{-0.106\,\bar{a}_x + 0.053\,\bar{a}_y\ \ A/m}$

Example 7.3 : *Find the magnetic flux density at the centre 'O' of a square of sides equal to 5 m and carrying 10 amperes of current.*

Solution : The square is placed in the xy plane as shown in the Fig. 7.19.

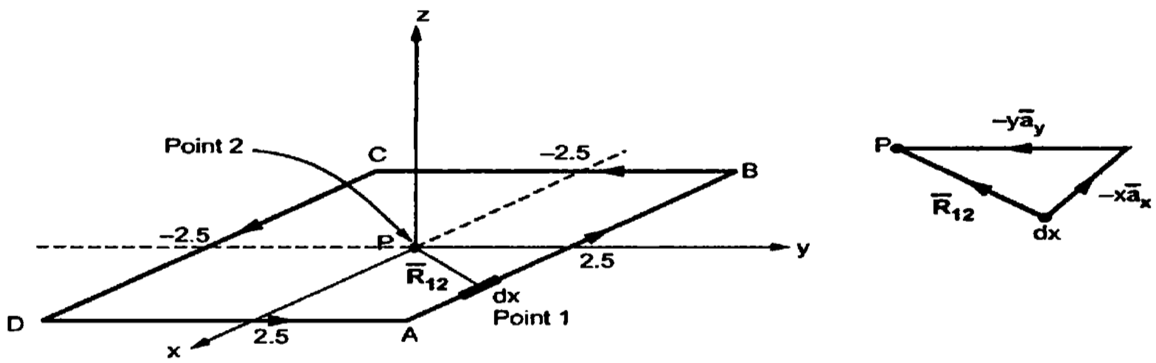


Fig. 7.19

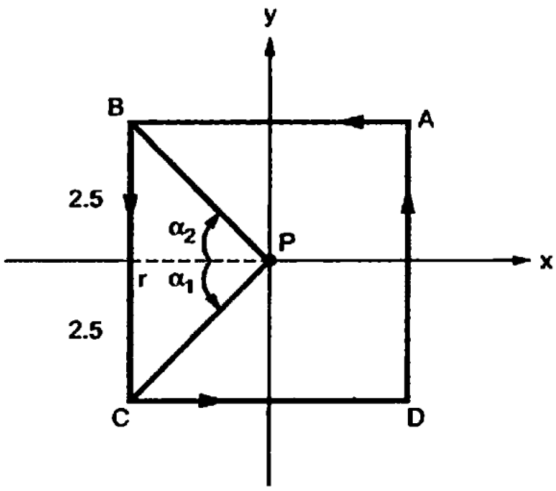


Fig. 7.20

Alternative method : Consider one side of a square as shown in the Fig. 7.20, in xy plane. Consider segment BC, which is finite length of the conductor. As B is above P, α_1 is negative and α_2 is positive.

$$\alpha_1 = \tan^{-1}\frac{2.5}{2.5} = 45°, \text{ but}$$

$$\alpha_1 = -45°$$

$$\alpha_2 = +45°$$

$$\therefore \quad |\bar{H}| = \frac{I}{4\pi r}\sin\alpha_2 - \sin\alpha_1 = \frac{10}{4\pi\times 2.5}[\sin(45°)-\sin(-45°)]$$

$$= 0.4501 \text{ A/m}$$

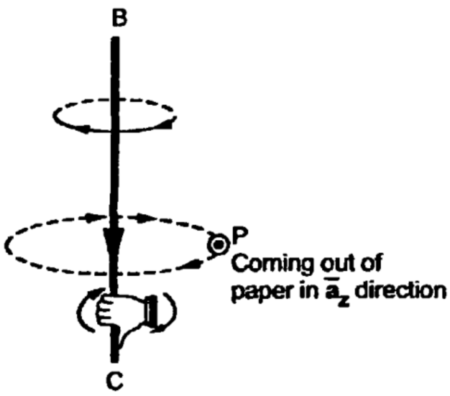


Fig. 7.21

Important note : As BC segment is not along z-axis while using formula derived earlier do not use direction as $\bar{a}_\phi$. Remember that $\bar{H}$ direction is normal to the plane containing the source. In this case, square is in xy plane normal to which is $\bar{a}_z$ hence direction of $\bar{H}$ is $\bar{a}_z$, as shown in the Fig. 7.21 by right handed screw rule.

$$\therefore \quad \bar{H} = 0.4501\ \bar{a}_z \text{ A/m}$$

$$\therefore \quad \bar{H}_{total} = 4\bar{H} = 1.8\ \bar{a}_z \text{ A/m}$$

3.3 Ampere's Circuital Law

In electrostatics, the Gauss's law is useful to obtain the $\vec{E}$ in case of complex problems. Similarly, in the magnetostatics, the complex problems can be solved using a law called **Ampere's circuital law** or **Ampere's work law**.

The Ampere's circuital law states that,

The line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

The mathematical representation of Ampere's circuital law is,

$$\oint_L \vec{H}.d\vec{L} = I_{enc} \ldots\ldots\ldots (1)$$

The law is very helpful to determine $\vec{H}$ when the current distribution is symmetrical.

By applying Stoke's theorem to left side of equation (1), we get

$$\oint_L \vec{H}.d\vec{L} = \int_S (\nabla \times \vec{H}).d\vec{S} \ldots\ldots (2)$$

$$\text{But } I_{enc} = \int_S \vec{J}.d\vec{S} \text{......... (3)}$$

Comparing equation (2) and (3), reveals that

$$\nabla \times \vec{H} = \vec{J} \text{........ (4)}$$

This is one of the Maxwell's equations.

3.3.1 Proof of Ampere's Circuital Law

Consider a long straight conductor carrying direct current I placed along z axis as shown in the Fig. 3.3.1. Consider a closed circular path of radius ρ which encloses the straight conductor carrying direct current I. The point P is at a perpendicular distance ρ from the conductor. Consider $d\vec{L}$ at point P which is in $\vec{a}_\phi$ direction, tangential to circular path at point P.

$$\therefore \qquad d\vec{L} = \rho d\phi \vec{a}_\phi \text{......... (5)}$$

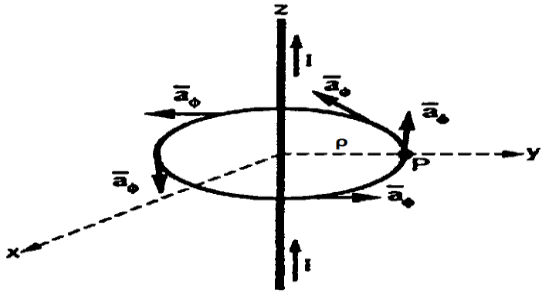


Fig. 3.3.1

While $\vec{H}$ obtained at point P, from Biot-Savart law due to infinitely long conductor is,

$$\vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi \text{....... (6)}$$

$$\therefore \qquad \vec{H}.d\vec{L} = \frac{I}{2\pi\rho}\vec{a}_\phi.\rho d\phi \vec{a}_\phi = \frac{I}{2\pi}d\phi$$

Integrating $\vec{H}.d\vec{L}$ over the entire closed path,

$$\oint_L \vec{H}.d\vec{L} = \int_{\phi=0}^{\phi=2\pi} \frac{I}{2\pi}d\phi = \frac{I}{2\pi}[\phi]\begin{matrix}2\pi\\0\end{matrix}$$

$$\oint_L \vec{H}.d\vec{L} = \frac{I2\pi}{2\pi} = I \qquad \text{Current carried by conductor}$$

This proves that the integral $\vec{H}.d\vec{L}$ along the closed path gives the direct current enclosed by that closed path.

Key Point: The path enclosing the direct current need not be a circular and it may be any irregular shape. The law does not depend on the shape of the path but the path must enclose the direct current once. This path selected is called **Amperian path** similar to the Gaussian surface used while applying Gauss's law.

3.3.2 Steps to Apply Ampere's Circuital Law

Follow the steps given to apply Ampere's circuital law:

Step 1: Consider a closed path preferably symmetrical such that it encloses the

Direct current I once. This is Amperian path.

Step 2: Consider differential length $d\vec{L}$ depending upon the co-ordinate system used.

Step3: Identify the symmetry and find in which direction $\vec{H}$ exists according to the co-ordinate system used.

Step 4: Find $\vec{H}.d\vec{L}$ the dot product. Make sure that $d\vec{L}$ and $\vec{H}$ in same direction

Step 5: Find the integral of $\vec{H}.d\vec{L}$ around the closed path assumed. And equate it to current I enclosed by the path.

Solving this for the $\vec{H}$ we get the required magnetic field intensity due to the direct current I.

To apply Ampere's circuital law, the following conditions must be satisfied,

1. The $\vec{H}$ is either tangential or normal to the path, at each point of the closed path.

2. The magnitude of $\vec{H}$ must be same at all points of the path where $\vec{H}$ is tangential.

Thus identifying symmetry and identifying the components of $\vec{H}$ present, plays an important role while applying the Ampere's circuital law.

3.3.3Applications of Ampere's Circuital Law

Let us steady the various cases and the application of Ampere's circuital law to obtain $\vec{H}$.

3.3.3.1 $\vec{H}$ due to Infinitely Long Straight Conductor

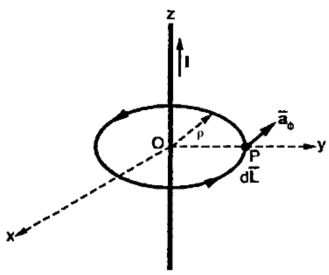


Fig. 3.3.2

Consider an infinitely long straight conductor placed along z-axis, carrying a direct current I as shown in the Fig.3.3.2. Consider the Amperian closed path, enclosing the conductor as shown in the Fig.3.3.2. Consider point P on the closed path at which $\vec{H}$ is to be obtained. The radius of the path is ρ and hence P is at a perpendicular distance ρ from the conductor.

The magnitude of $\vec{H}$ depends on ρ and the direction is always tangential to the closed path i.e. $\vec{a}_\phi$. So $\vec{H}$ has only component in $\vec{a}_\phi$ direction.

$$\therefore \qquad \vec{H} = H_\phi \vec{a}_\phi \text{ and } d\vec{L} = \rho d\phi \vec{a}_\phi$$

$$\therefore \qquad \vec{H}.d\vec{L} = H_\phi \rho d\phi$$

According to Ampere's circuital law,

$$\oint_L \vec{H}.d\vec{L} = I$$

$$\therefore \quad \int_{\phi=0}^{\phi=2\pi} H_\phi \rho d\phi = I$$

$$\therefore \; H_\phi \rho 2\pi = I$$

$$\therefore \; H_\phi = \frac{I}{2\pi\rho}$$

Hence $\vec{H}$ at point P is given by,

$$\vec{H} = H_\phi \vec{a}_\phi = \frac{I}{2\pi\rho}\vec{a}_\phi \text{ A/m....... (7)}$$

3.3.3.2 $\vec{H}$ due to a Co-axial Cable

Consider a co-axial cable as shown in the Fig.3.3.3. Its inner conductor is solid with radius ‘a’, carrying direct current I. The outer conductor is in the form of concentric cylinder whose inner radius is ‘b’ and outer radius is ‘c’. This cable is placed along z axis. The current I is uniformly distributed in the inner conductor. While - I is uniformly distributed in the outer conductor.

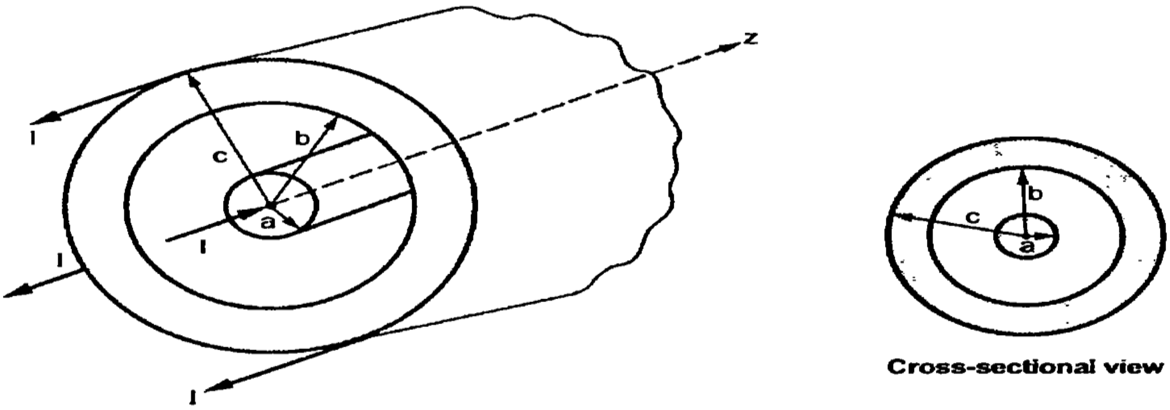


Fig. 3.3.3 Co-axial cable

The space between inner and outer conductor is filled with dielectric say air. The calculation of $\vec{H}$ is divided corresponding to various regions of the cable.

Region 1: Within the inner conductor, ρ< a.

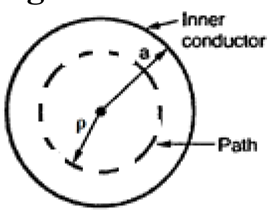


Fig. 3.3.4

Consider a closed path having radius ρ < a. Hence it encloses only part of the conductor as shown in the Fig.3.3.4. The area of cross section enclosed is $\pi\rho^2$.The total current flowing is I through the area πa^2. Hence the current enclosed by the closed path is,

$$I' = \frac{\pi\rho^2}{\pi a^2} I = \frac{\rho^2}{a^2} I \text{...... (8)}$$

The $\vec{H}$ and $d\vec{L}$ are again only $\vec{a}_\phi$ direction and depends only on ρ.

$$\therefore \qquad \vec{H} = H_\phi \vec{a}_\phi \text{ and } d\vec{L} = \rho d\phi \vec{a}_\phi$$

$\therefore \qquad \vec{H}.d\vec{L} = H_{\phi}\rho d\phi$

According to Ampere's circuital law,

$$\oint_L \vec{H}.d\vec{L} = I_{enc} = I'$$

$$\therefore \quad \int_{\phi=0}^{\phi=2\pi} H_{\phi}\rho d\phi = \frac{\rho^2}{a^2}I$$

$$\therefore \; H_{\phi}\rho 2\pi = \frac{\rho^2}{a^2}I$$

$$\therefore \; H_{\phi} = \frac{I\rho}{2\pi a^2}$$

Hence $\vec{H}$ at point P is given by,

$$\vec{H} = H_{\phi}\vec{a}_{\phi} = \frac{I\rho}{2\pi a^2}\vec{a}_{\phi} \text{ A/m……. (9)}$$

Region2: Within $a < \rho < b$

Consider a circular path which encloses the inner conductor carrying direct current I. This is the case of infinitely long conductor along z-axis. Hence $\vec{H}$ in this region is,

$$\vec{H} = \frac{I}{2\pi\rho}\vec{a}_{\phi} \text{ A/m……. (10)}$$

Region3: Within $b < \rho < c$

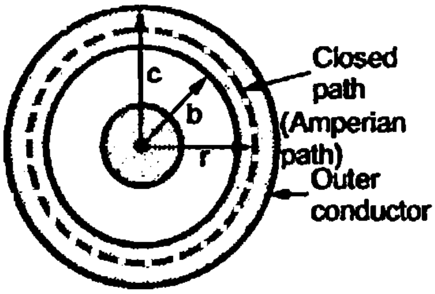


Fig.3.3.5

Consider a circular path as shown in Fig. 3.3.5. The current enclosed by the closed path is only the part of current $-I$, in the outer conductor. The total conductor $-I$ is flowing through the cross section $\pi(c^2 - b^2)$ while the closed path encloses the cross section $\pi(\rho^2 - b^2)$.

Hence the current enclosed by the closed path of outer conductor is,

$$I' = \frac{\pi(\rho^2-b^2)}{\pi(c^2-b^2)}(-I) = -\frac{(\rho^2-b^2)}{(c^2-b^2)}I \text{…… (11)}$$

Key Point: Note that the closed path also encloses the inner conductor hence the current I flowing through it.

$\therefore \qquad I'' = I =$Current in inner conductor enclosed …... (12)

Total current enclosed by the closed path is,

$$I = I' + I'' = -\frac{(\rho^2 - b^2)}{(c^2 - b^2)}I + I$$

$$= I\left[1 - \frac{(\rho^2 - b^2)}{(c^2 - b^2)}\right] = I\left[\frac{(c^2 - \rho^2)}{(c^2 - b^2)}\right]$$

The $\vec{H}$ and $d\vec{L}$ are again only $\vec{a}_{\phi}$ direction and depends only on ρ.

$\therefore \quad \vec{H} = H_\phi \vec{a}_\phi$ and $d\vec{L} = \rho d\phi \vec{a}_\phi$

$\therefore \quad \vec{H}.d\vec{L} = H_\phi \rho d\phi$

According to Ampere's circuital law,

$$\oint_L \vec{H}.d\vec{L} = I_{enc}$$

$$\therefore \quad \int_{\phi=0}^{\phi=2\pi} H_\phi \rho d\phi = \left[\frac{(c^2-\rho^2)}{(c^2-b^2)}\right] I$$

$$\therefore \; H_\phi \rho 2\pi = \left[\frac{(c^2-\rho^2)}{(c^2-b^2)}\right] I$$

$$\therefore \; H_\phi = \left[\frac{(c^2-\rho^2)}{(c^2-b^2)}\right] \frac{I}{2\pi\rho}$$

Hence $\vec{H}$ at point P is given by,

$$\vec{H} = H_\phi \vec{a}_\phi = \frac{I}{2\pi\rho}\left[\frac{(c^2-\rho^2)}{(c^2-b^2)}\right] \vec{a}_\phi \text{ A/m……. (13)}$$

Region 4 : Outside the cable, ρ > c.

Consider the closed path with $\rho > c$ such that it encloses both the conductors i.e. both currents $+I$ and $-I$.

Thus the total current enclosed is,

$$I_{enc} = +I - I = 0A$$

$\therefore \quad \oint_L \vec{H}.d\vec{L} = I_{enc} = 0$ (as $\rho > c$) Ampere's circuital law

The magnetic field does not exist outside the cable. The variation of $\vec{H}$ against ρ is shown in the Fig.3.3.6.

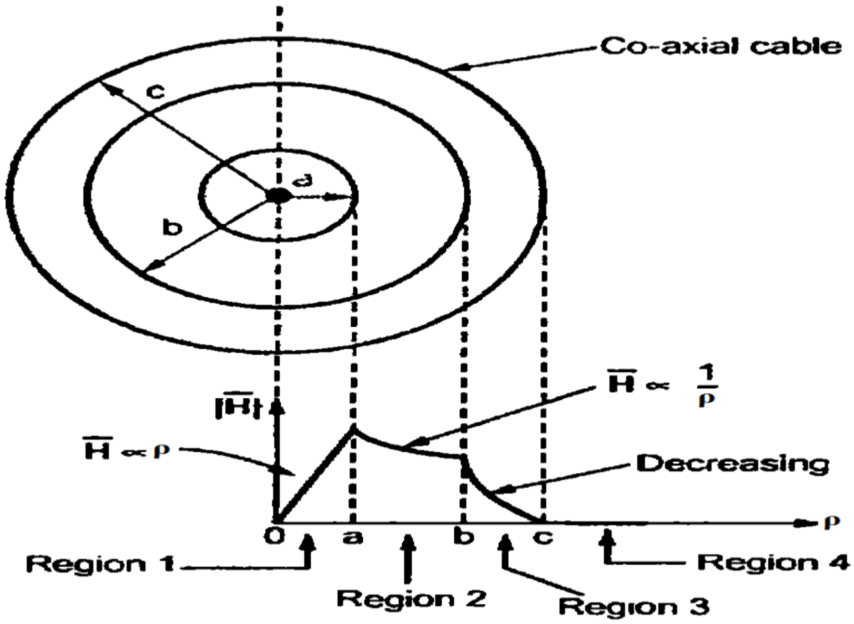


Fig.3.3.6 Variation of $\overline{H}$ against ρ in co-axial cable

3.3.3.3 $\vec{H}$ due to Infinite Sheet of Current

Consider an infinite sheet of current in the $z = 0$ plane. The surface current density is $\vec{K}$. The current is flowing in positive y direction hence $\vec{K} = K_y \vec{a}_y$. This is shown in the Fig.3.3.7.

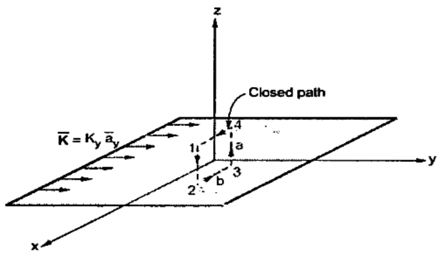


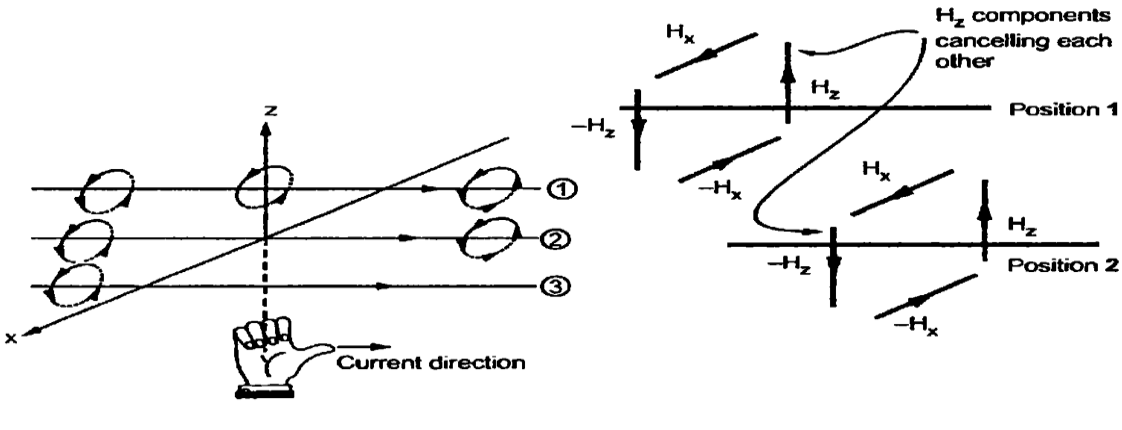
Fig.3.3.7

Consider a closed path 1-2-3-4 as shown in Fig.3.3.7. The width of the path is ‘b’ while the height is ‘a’. It is perpendicular to the direction of current hence in xz plane.

The current flowing across the distance b is given by $K_y b$.

$$I_{enc} = K_y b \ldots\ldots.. \quad (14)$$

Consider the magnetic lines of force due to the current in $\vec{a}_y$ direction, according to right hand thumb rule. These are shown in the Fig. 3.3.8.



(a) (b)

Fig. 3.3.8

In Fig.3.3.8 (b), it is clear that in between two very closely spaced conductors, the components of $\vec{H}$ in z direction are oppositely directed ($-H_z$, for position 1 and $+H_z$in for position 2 between the two positions). All such components cancel each other and hence $\vec{H}$ cannot have any component in z direction. **As current is flowing in y direction, $\vec{H}$ can not have component in y direction.**

So $\vec{H}$ has only component in x direction.

$$\therefore \quad \vec{H} = H_x \vec{a}_x \quad \ldots\text{for } z > 0 \quad \ldots\ldots\ldots(15(a))$$

$$= -H_x \vec{a}_x \quad \ldots\text{for } z < 0 \quad \ldots\ldots\ldots(15(b))$$

Applying Ampere’s circuit law,

$$\oint_L \vec{H}.d\vec{L} = I_{enc}$$

Evaluate the integral along the path 1-2-3-4-1.

For path 1-2, $d\vec{L} = dz\vec{a}_z$

For path 3-4, $d\vec{L} = dz\vec{a}_z$

But $\vec{H}$ is in x direction while $\vec{a}_x.\vec{a}_z = 0$

Hence along the paths 1-2 and 3-4, the integral $\oint_L \vec{H}.d\vec{L} = 0$.

Consider path 2-3 along which $d\vec{L} = dx\vec{a}_x$.

$\therefore \quad \int_2^3 \vec{H}.d\vec{L} = \int_2^3 -H_x\vec{a}_x.dx\vec{a}_x = H_x\int_2^3 dx = bH_x$

The path 2-3 is lying in $z < 0$ region for which $\vec{H}$ is $-H_x\vec{a}_x$. And limits from 2 to 3, positive x to negative x hence effective sign of the integral is positive.

Consider path 4-1 along which $d\vec{L} = dx\vec{a}_x$ and it is in the region $z > 0$ hence $\vec{H} = H_x\vec{a}_x$.

$\therefore \quad \int_4^1 \vec{H}.d\vec{L} = \int_4^1 H_x\vec{a}_x.dx\vec{a}_x = H_x\int_4^1 dx = bH_x$

$\therefore \quad \oint_L \vec{H}.d\vec{L} = bH_x + bH_x = 2bH_x$………… (16)

Equating this to I_{enc} in equation (14),

$$2bH_x = K_y b$$

$$\therefore \quad H_x = \frac{1}{2}K_y$$

Hence, $\vec{H} = \frac{1}{2}K_y\vec{a}_x$ for $z > 0$……….. (17(a))

$= -\frac{1}{2}K_y\vec{a}_x$ for $z < 0$……….. (17(b))

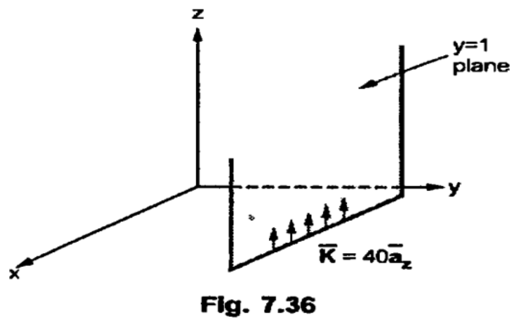
In general, for an infinite sheet of current density $\vec{K}$ A/m we can write,

$$\vec{H} = \frac{1}{2}\vec{K} \times \vec{a}_N \text{…… (18)}$$

Where $\vec{a}_N$ = Unit vector normal from the current sheet to the point at which $\vec{H}$ is to be obtained.

Example 7.6 : *The plane $y = 1$ carries current density $\bar{K} = 40\bar{a}_z$ A/m. Find H at A (0,0,0) and B(1,5,–2).*

Solution : The sheet is located at y = 1 on which $\bar{K}$ is in $\bar{a}_z$ direction. The sheet is infinite and is shown in the Fig. 7.36.



P

Fig. 7.36

The $\bar{H}$ will be in x direction.

a) Point A (0, 0, 0)

$\bar{a}_N = -\bar{a}_y$ normal to current sheet at Point A

$\therefore \quad \bar{H} = \frac{1}{2}\bar{K} \times \bar{a}_N$

$= \frac{1}{2}[40\bar{a}_z \times -\bar{a}_y]$

Now $\quad \bar{a}_z \times \bar{a}_y = -\bar{a}_x$

$\therefore \quad \bar{H} = \frac{1}{2}[+40]\bar{a}_x = 20\bar{a}_x$ A/m

b) Point B (1, 5, –2)

This is to the right of the plane as y = 5 for B.

$\therefore \quad \bar{a}_N = \bar{a}_y$ normal to sheet at point B

$\therefore \quad \bar{H} = \frac{1}{2}\bar{K}\times\bar{a}_N = \frac{1}{2}[40\bar{a}_z \times \bar{a}_y] = -20\bar{a}_x$ A/m

Example 7.7 : *In the region $0 < r < 0.5$ m, in cylindrical co-ordinates, the current density is,*

$\bar{J} = 4.5\, e^{-2r}\, \bar{a}_z\ A/m^2$

And $\bar{J} = 0$ elsewhere. Use Amperes circuital law to find $\bar{H}$.

Solution : The current from current density is given by,

$$I = \oint \bar{J} \cdot d\bar{S}$$

$$d\bar{S} = r\, dr\, d\phi\, \bar{a}_z \quad \text{normal to } \bar{a}_z \text{ as } \bar{J} \text{ is in } \bar{a}_z$$

$$\therefore \quad I = \int_{\phi=0}^{2\pi} \int_{r=0}^{r} 4.5\, e^{-2r}\, \bar{a}_z \cdot r\, dr\, d\phi\, \bar{a}_z$$

$$= 4.5 \int_{\phi=0}^{2\pi} \int_{r=0}^{r} r\, e^{-2r}\, dr\, d\phi$$

Using integration by parts,

$$= 4.5 \int_{\phi=0}^{2\pi} d\phi \left\{ r \int e^{-2r}\, dr - \int 1 \int e^{-2r}\, dr\, dr \right\}_0^r$$

$$= 4.5\,(2\pi) \left\{ \frac{r\, e^{-2r}}{-2} - \int \frac{e^{-2r}}{-2}\, dr \right\}_0^r$$

$$= 9\pi \left\{ \frac{r\, e^{-2r}}{-2} + \frac{1}{2}\frac{e^{-2r}}{-2} \right\}_0^r$$

$$= 9\pi \left\{ \frac{-r\, e^{-2r}}{2} - \frac{1}{4} e^{-2r} + \frac{1}{4} \right\} = \frac{9\pi}{4} \left\{ 1 - 2r\, e^{-2r} - e^{-2r} \right\} \text{ A}$$

For $r = 0.5$, $I = 7.068\,[1 - 0.3678 - 0.3678] = 1.8676$ A

Consider a closed path with $r \geq 0.5$ such that the enclosed current I is 1.8676 A.

According to Ampere's circuital law,

$$\oint \bar{H} \cdot d\bar{L} = I$$

$$\therefore \quad \int_{\phi=0}^{2\pi} H_\phi\, r\, d\phi = I \qquad \because \bar{H} = H_\phi\, \bar{a}_\phi,\ d\bar{L} = r\, d\phi\, \bar{a}_\phi$$

$$\therefore \quad 2\pi r H_\phi = 1.8676$$

$$\therefore \quad H_\phi = \frac{1.8676}{2\pi r} = \frac{0.2972}{r}$$

$$\therefore \quad \bar{H} = \frac{0.2972}{r}\, \bar{a}_\phi \quad \text{A/m} \quad \text{for } r \geq 0.5 \text{ m}$$

3.4 Magnetic Flux and Flux Density

The magnetic flux density $\vec{B}$ is analogous to the electric flux density $\vec{D}$. The relation between $\vec{B}$ and $\vec{H}$ is already mentioned, which is through the property of medium called permeability μ. The relation is given by,

$$\vec{B} = \mu\vec{H} \ldots\ldots (1)$$

For the free space, $\mu = \mu_0 = 4\pi \times 10^{-7}$H/m hence,

$$\vec{B} = \mu_0\vec{H} \quad \text{for free space} \ldots\ldots\ldots (2)$$

The magnetic flux density has units Wb/m^2 and hence it can be defined as the flux in webers passing through unit area in a plane at right angle to the direction of flux.

If the flux passing through the unit area is not exactly at right angles to the plane consisting the area but making some angle with the plane then the flux ϕ crossing the area is given by,

$$\phi = \int_S \vec{B}.d\vec{S} \text{ weber } \ldots\ldots (3)$$

where $\phi =$ Magnetic flux in webers

$\vec{B} =$ Magnetic flux density in Wb/m^2 or Tesla (T)

$d\vec{S} =$Open surface through which flux is passing.

Now consider a closed surface which is defining a certain volume. The magnetic flux lines are always exist in the form of closed loop. Thus for a closed surface the number of magnetic flux lines entering must be equal to the number of magnetic flux lines leaving. The single magnetic pole cannot exist like a single isolated charge. No magnetic flux can reside in a closed surface. Hence the integral $\vec{B}.d\vec{S}$ evaluated over a closed surface is always zero.

$$\therefore \quad \oint_S \vec{B}.d\vec{S} = 0 \ldots\ldots (4)$$

This is called **law of conservation of magnetic flux or Gauss's law in integral form for magnetic fields**.

Applying divergence theorem to equation (4),

$$\oint_S \vec{B}.d\vec{S} = \int_v (\nabla.\vec{B}) dv \ldots\ldots\ldots (5)$$

where dv = Volume enclosed by the closed surface.

But as dv is not zero, we can write,

$$\nabla.\vec{B} = 0 \ldots\ldots\ldots (6)$$

The divergence of magnetic flux density is always zero. This is called **Gauss's law in differential form for magnetic fields**. This is another Maxwell's equation.

Example 7.15 : *In cylindrical co-ordinates $\bar{B} = (2.0/r)\bar{a}_\phi$ Tesla. Determine the magnetic flux ϕ crossing the plane surface defined by $0.5 \le r \le 2.5$ m and $0 \le z \le 2$ m.*

Solution : The surface is shown in the Fig. 7.42.

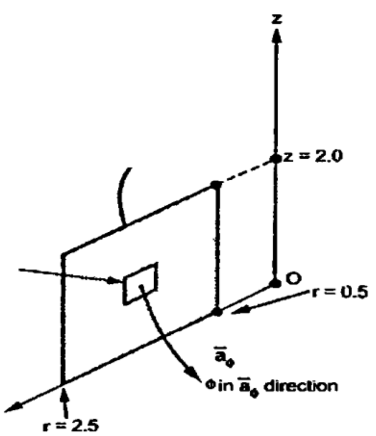


Fig. 7.42

The flux crossing the surface is given by,

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

The $d\bar{S}$ normal to $\bar{a}_\phi$ direction is dr dz.

$$\therefore \quad d\bar{S} = dr\, dz\, \bar{a}_\phi$$

$$\therefore \quad \phi = \int_S \frac{2.0}{r} \bar{a}_\phi \cdot dr\, dz\, \bar{a}_\phi$$

$$= \int_{z=0}^{2} \int_{r=0.5}^{2.5} \frac{2.0}{r}\, dr\, dz$$

$$= 2.0\, [ln\, r]_{0.5}^{2.5}\, [z]_0^2$$

$$= 2.0\, [ln\, 2.5 - ln\, 0.5]\, [2 - 0]$$

$$= 6.4377 \text{ Wb}$$

3.5 Maxwell's Equations for Static Electromagnetic Fields

Let us summarize the Maxwell's equations for static electric and magnetic fields.

Maxwell's equations in differential or point form	
1. $\nabla . \vec{D} = \rho_v$	Gauss's law
2. $\nabla \times \vec{E} = 0$	Conservation of electric field
3. $\nabla \times \vec{H} = \vec{J}$	Ampere's circuital law
4. $\nabla . \vec{B} = 0$	Single magnetic pole cannot exist i.e. conservation of magnetic flux.

Table 3.1

The Maxwell's equations in integral form an be summarized as,

Maxwell's equations in integral form
1. $\oint_S \vec{D} . d\vec{S} = \int_v \rho_v \, dv = Q$
2. $\oint_L \vec{E} . d\vec{L} = 0$
3. $\oint_L \vec{H} . d\vec{L} = \int_S \vec{J} . d\vec{S} = I$
4. $\oint_S \vec{B} . d\vec{S} = 0$

Table 3.2

3.6 Magnetic Scalar and Vector Potentials

In electrostatics, it is seen that there exists a scalar electric potential V which is related to the electric field intensity $\vec{E}$ as $\vec{E} = -\nabla V$.

Is there any scalar potential in magnetostatics related to magnetic field intensity $\vec{H}$?

In case of magnetic fields there are two types of potentials which can be defined:

1. The scalar magnetic potential denoted as V_m.
2. The vector magnetic potential denoted as $\vec{A}$.

To define scalar and vector magnetic potentials, let us use two vector identities which are listed as the properties of curl, earlier.

$$\nabla \times \nabla V = 0, \text{ V = Scalar} \ldots\ldots\ldots\ldots \text{ (1)}$$

$$\nabla . \left(\nabla \times \vec{A}\right) = 0, \vec{A} = \text{Vector} \ldots\ldots\ldots\ldots \text{ (2)}$$

Every Scalar V and Vector $\vec{A}$ must satisfy these identities.

3.6.1 Scalar Magnetic Potential

If V_m is the scalar magnetic potential then it must satisfy the equation (1),

$\therefore$
$$\nabla \times \nabla V_m = 0 \ldots\ldots\ldots\ldots \text{ (3)}$$

But the scalar magnetic potential is related to the magnetic field intensity $\vec{H}$ as,

$$\vec{H} = -\nabla V_m \ldots\ldots\ldots\ldots \text{ (4)}$$

Using in equation (3),

$$\nabla \times -\vec{H} = 0 \qquad \text{i.e. } \nabla \times \vec{H} = 0 \ \ldots\ldots\ldots (5)$$

But $$\nabla \times \vec{H} = \vec{J} \ \text{ i.e. } \vec{J} = 0 \ldots\ldots\ldots (6)$$

Thus **scalar magnetic potential V_m can be defined for source free region where $\vec{J}$ i.e. current density is zero**.

∴ $$\vec{H} = -\nabla V_m \quad \text{only for } \vec{J} = 0 \ldots\ldots\ldots (7)$$

Similar to the relation between $\vec{E}$ and electric scalar potential, magnetic scalar potential can be expressed in terms of $\vec{H}$ as,

$$V_{m\,a,b} = -\int_a^b \vec{H}.d\vec{L} \ \ldots\ldots \text{ Specific path}$$

3.6.2 Laplace's Equation for Scalar Magnetic Potential

It is known that as monopole of magnetic field is non existing,

$$\oint_S \vec{B}.d\vec{S} = 0 \ldots\ldots\ldots (8)$$

Using Divergence theorem,

$$\oint_S \vec{B}.d\vec{S} = \int_v (\nabla.\vec{B})dv = 0$$

∴ $$\nabla.\vec{B} = 0 \quad \text{as} \quad dv \neq 0 \ldots\ldots\ldots (9)$$

∴ $$\nabla.(\mu_0 \overrightarrow{H}) = 0 \quad \text{but} \quad \mu_0 \neq 0$$

∴ $$\nabla.\vec{H} = 0$$

∴ $$\nabla.(-\nabla V_m) = 0$$

∴ $$\nabla^2 V_m = 0 \qquad \text{for } \vec{J} = 0 \ldots\ldots\ldots (10)$$

This is Laplace's equation for scalar magnetic potential. This is similar to the Laplace's equation for scalar electric potential $\nabla^2 V = 0$.

3.6.3 Vector Magnetic Potential

The vector magnetic potential is denoted as $\vec{A}$ and measured in Wb/m. lt has to satisfy equation (2) that divergence of a curl of a vector is always zero.

∴ $$\nabla.\left(\nabla \times \vec{A}\right) = 0, \quad \vec{A} = \text{Vector magnetic potential}$$

But $$\nabla.\vec{B} = 0 \quad \text{... from equation (9)}$$

∴ $$\vec{B} = \nabla \times \vec{A} \ldots\ldots\ldots (11)$$

Thus **curl of vector magnetic potential is the flux density.**

Now $$\nabla \times \vec{H} = \vec{J}$$

∴ $$\nabla \times \frac{\vec{B}}{\mu_0} = \vec{J} \qquad \text{as} \qquad \vec{B} = \mu_0 \vec{H}$$

∴ $$\nabla \times \vec{B} = \mu_0 \vec{J}$$

∴ $$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} \qquad \text{as } \vec{B} = \nabla \times \vec{A}$$

Using vector identity to express left hand side we can write,

$$\nabla\left(\nabla.\vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J} \ldots\ldots\ldots (12)$$

$\therefore \qquad \vec{J} = \frac{1}{\mu_0}\left[\nabla \times \nabla \times \vec{A}\right] = \frac{1}{\mu_0}\left[\nabla(\nabla.\vec{A}) - \nabla^2\vec{A}\right] \ldots\ldots\ldots (13)$

Thus if vector magnetic potential is known then current density $\vec{J}$ can be obtained. For defining $\vec{A}$ the current density need not be zero.

3.6.4 Poisson's Equation for Magnetic Field

In a vector algebra, a vector can be fully defined if its curl and divergence are defined. For a vector magnetic potential $\vec{A}$, its curl is defined as $\nabla \times \vec{A} = \vec{B}$ which is known.

But to completely define $\vec{A}$ its divergence must be known. Assume that $\nabla.\vec{A}$, the divergence of $\vec{A}$ is zero. This is consistent with some other conditions to be studied later in time varying magnetic fields. Using in equation (13),

$$\vec{J} = \frac{1}{\mu_0}\left[-\nabla^2\vec{A}\right]$$

$$\nabla^2\vec{A} = -\mu_0\vec{J} \ldots\ldots\ldots (14)$$

This is the Poisson's equation for magnetostatic fields.

3.6.5 $\vec{A}$ due to Differential Current Element

Consider the differential element $d\vec{L}$ carrying current I. Then according to Biot-Savart law the vector magnetic potential $\vec{A}$ at a distance R from the differential current element is given by,

$$\vec{A} = \oint_L \frac{\mu_0 I d\vec{L}}{4\pi R} \text{ Wb/m} \ldots\ldots\ldots (15)$$

For the distributed current sources, $Id\vec{L}$ can be replaced by $\vec{K}dS$ where $\vec{K}$ is surface current density.

$$\vec{A} = \oint_S \frac{\mu_0 \vec{K} dS}{4\pi R} \text{ Wb/m} \ldots\ldots\ldots (16)$$

The line integral becomes a surface integral. If the volume current density $\vec{J}$ is given in A/m^2 then $Id\vec{L}$ can be replaced by $\vec{J}dv$ where dv is differential volume element.

$$\vec{A} = \oint_v \frac{\mu_0 \vec{J} dv}{4\pi R} \text{ Wb/m} \ldots\ldots\ldots (17)$$

It can be noted that,

1. The zero reference for $\vec{A}$ is at infinity.
2. No finite current can produce the contributions as $R \to \infty$.

Example 7.17 : *In cylindrical co-ordinates $\bar{A} = 50r^2\,\bar{a}_z$ Wb/m is a vector magnetic potential, in a certain region of free space. Find, $\bar{H}, \bar{B}, \bar{J}$ and using $\bar{J}$ find total current I crossing the surface $0 \le r \le 1$, $0 \le \phi \le 2\pi$ and $z = 0$.*

Solution : Vector magnetic potential, $\bar{A} = 50\,r^2\,\bar{a}_z$ Wb/m

Now, $\qquad \bar{B} = \nabla \times \bar{A}$

$$= \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\bar{a}_\phi + \frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right]\bar{a}_z$$

Now $A_r = 0, \quad A_\phi = 0, \quad A_z = 50\, r^2$

$$\therefore \quad \bar{B} = \left[\frac{1}{r}\frac{\partial(50\,r^2)}{\partial\phi} - 0\right]\bar{a}_r + \left[0 - \frac{\partial(50\,r^2)}{\partial r}\right]\bar{a}_\phi + \frac{1}{r}[0-0]\bar{a}_z$$

$$\bar{B} = -100\, r\bar{a}_\phi \quad \text{Wb/m}^2$$

$$\therefore \quad \bar{H} = \frac{\bar{B}}{\mu_0} = \frac{-100}{\mu_0} r\, a_\phi \quad \text{A/m}$$

Now $\bar{J} = \nabla \times \bar{H}$

$$H_r = 0, \quad H_\phi = -\frac{100\, r}{\mu_0}, \quad H_z = 0$$

$$\therefore \quad \nabla \times \bar{H} = \left[0 - \frac{\partial\left(-\frac{100\,r}{\mu_0}\right)}{\partial z}\right]\bar{a}_r + [0-0]\bar{a}_\phi + \frac{1}{r}\left[\frac{\partial\left(-\frac{100\,r^2}{\mu_0}\right)}{\partial r} - 0\right]\bar{a}_z$$

$$= [0-0]\bar{a}_r + 0\,\bar{a}_\phi + \frac{1}{r}\left[-\frac{100}{\mu_0}\right][2r]\,\bar{a}_z = -\frac{200}{\mu_0}\bar{a}_z \quad \text{A/m}^2$$

$$\therefore \quad \bar{J} = -\frac{200}{\mu_0}\bar{a}_z$$

Now $I = \int_S \bar{J}\cdot d\bar{S}$ where $d\bar{S} = r\, dr\, d\phi\, \bar{a}_z$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{1} -\frac{200}{\mu_0}\bar{a}_z \cdot r\, dr\, d\phi\, \bar{a}_z = \int_{\phi=0}^{2\pi}\int_{r=0}^{1} -\frac{200}{\mu_0} r\, dr\, d\phi$$

$$= -\frac{200}{\mu_0}\left[\frac{r^2}{2}\right]_0^1 [\phi]_0^{2\pi} = \frac{-200}{\mu_0}\left[\frac{1}{2}\right][2\pi]$$

$$= -500 \times 10^6 \text{ A}$$

So current is 500 MA and negative sign indicates the direction of current.

3.7 Force on a Moving Point Charge

According to the discussion in the previous chapters, a static electric field $\vec{E}$ exerts a force on a static or moving charge Q. Thus according to Coulomb's law, the force $\vec{F}_e$ exerted on an electric charge can be obtained. The force is related to the electric field intensity $\vec{E}$ as,

$$\vec{F}_e = Q\vec{E} \quad \text{Newton} \; .... (1)$$

For a positive charge, the force exerted on it is in the direction of $\vec{E}$. This force is also referred as **electric force** ($\vec{F}_e$).

Now consider that a charge is placed in a steady magnetic field. It experiences a force only if it is moving. Then a magnetic force ($\vec{F}_m$) exerted on a charge Q, moving with a velocity $\vec{v}$ in a steady magnetic field $\vec{B}$ is given by,

$$\vec{F}_m = Q(\vec{v} \times \vec{B}) \quad \text{Newton} \; .... (2)$$

The magnitude of the magnetic force $\vec{F}_m$ is directly proportional to the magnitudes of Q, $\vec{v}$ and $\vec{B}$ and also the sine of the angle between $\vec{v}$ and $\vec{B}$. The direction of $\vec{F}_m$ is perpendicular to the plane containing $\vec{v}$ and $\vec{B}$ both, as shown in the Fig. 3.7.1.

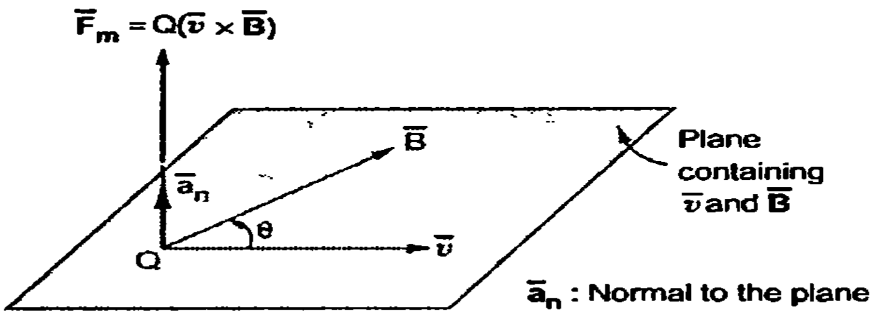


Fig. 3.7.1 Magnetic force on a moving charge in magnetic field

The total force on a moving charge in the presence of both electric and magnetic fields is given by,

$$\vec{F} = \vec{F}_e + \vec{F}_m = Q(\vec{E} + \vec{v} \times \vec{B}) \text{ Newton...... (3)}$$

Above equation is called **Lorentz Force Equation** which relates mechanical force to the electrical force. If the mass of the charge is m, then we can write,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = Q(\vec{E} + \vec{v} \times \vec{B}) \text{ Newton...... (4)}$$

Example 8.1 *A point charge of Q = − 1.2 C has velocity $\bar{v} = (5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z)$ m/s. Find the magnitude of the force exerted on the charge if,*

a) $E = -18\bar{a}_x + 5\bar{a}_y - 10\bar{a}_z$ V/m,

b) $\bar{B} = -4\bar{a}_x + 4\bar{a}_y + 3\bar{a}_z$ T,

c) Both are present simultaneous.

Solution : a) The electric force exerted by $\bar{E}$ on charge Q is given by,

$$\bar{F}_e = Q\bar{E}$$
$$= -1.2[-18\bar{a}_x + 5\bar{a}_y - 10\bar{a}_z]$$
$$= 21.6\,\bar{a}_x - 6\,\bar{a}_y + 12\,\bar{a}_z \text{ N}$$

Thus the magnitude of the electric force is given by

$$|\bar{F}_e| = \sqrt{(21.6)^2 + (-6)^2 + (12)^2} = 25.4275 \text{ N}$$

b) The magnetic force exerted by $\bar{B}$ on charge Q is given by,

$$\bar{F}_m = Q\,\bar{v} \times \bar{B}$$
$$= -1.2[(5\bar{a}_x + 2\bar{a}_y - 3\bar{a}_z) \times (-4\bar{a}_x + 4\bar{a}_y + 3\bar{a}_z)]$$
$$= (-6\,\bar{a}_x - 2.4\bar{a}_y + 3.6\bar{a}_z) \times (-4\bar{a}_z + 4\,\bar{a}_y + 3\,\bar{a}_z)$$
$$= \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ -6 & -2.4 & 3.6 \\ -4 & 4 & 3 \end{vmatrix}$$
$$= [-7.2 - 14.4]\,\bar{a}_x - [-18 + 14.4]\,\bar{a}_y + [-24 - 9.6]\,\bar{a}_z$$
$$= (-21.6\,\bar{a}_x + 3.6\,\bar{a}_y - 33.6\,\bar{a}_z) \text{ N}$$

Thus, the magnitude of the magnetic force is given by

$$|\bar{F}_m| = \sqrt{(-21.6)^2 + (+3.6)^2 + (-33.6)^2} = 40.1058 \text{ N}$$

c) The total force exerted by both the fields ($\bar{E}$ and $\bar{B}$) on a charge is given by,

$$\begin{aligned} \bar{F} &= \bar{F}_e + \bar{F}_m = Q\left(\bar{E} + \bar{v} \times \bar{B}\right) = Q\,\bar{E} + Q\,\bar{v} \times \bar{B} \\ &= \left[\left(21.6\,\bar{a}_x - 6\,\bar{a}_y + 12\,\bar{a}_z\right) + \left(-21.6\,\bar{a}_x + 3.6\,\bar{a}_y - 33.6\,\bar{a}_z\right)\right] \\ &= \left(0\,\bar{a}_x - 2.4\,\bar{a}_y - 21.6\,\bar{a}_z\right)\ \text{N} \end{aligned}$$

Thus, the magnitude of the total force exerted is given by

$$|\bar{F}| = \sqrt{(0)^2 + (-2.4)^2 + (-21.6)^2} = 21.7329\ \text{N}$$

3.8 Magnetic Boundary Conditions

The conditions of the magnetic field existing at the boundary of the two media when the magnetic field passes from one medium to other are called **boundary conditions for magnetic fields** or simply **magnetic boundary conditions**. To study conditions of $\vec{B}$ and $\vec{H}$ at the boundary, both the vectors are resolved into two components ;

a) Tangential to boundary and

b) Normal (perpendicular) to boundary.

Consider a boundary between two isotropic, homogeneous linear materials with different permeabilities μ_1 and μ_2 as shown in the Fig.3.8.1. To determine the boundary conditions, let us use the closed path and the Gaussian surface.

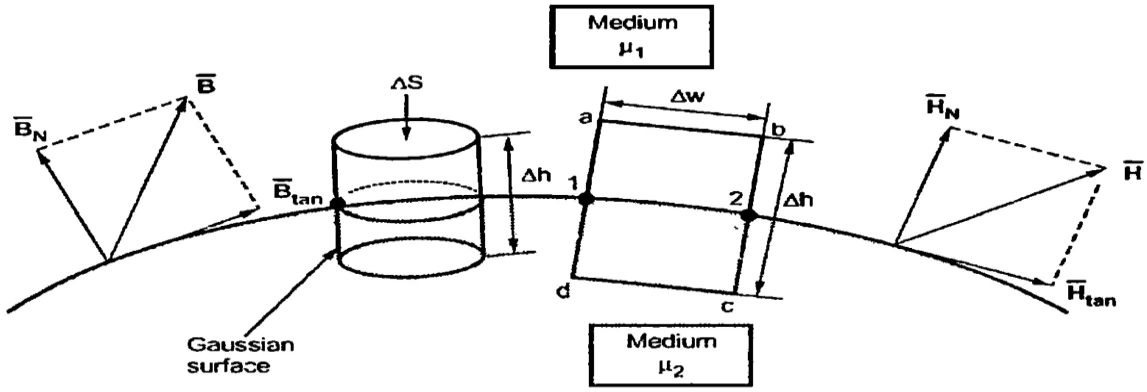


Fig. 3.8.1 Boundary between two magnetic materials of different permeabilities

3.8.1 Boundary Conditions for Normal Component

To find the normal component of $\vec{B}$, select a closed Gaussian surface in the form of a right circular cylinder as shown in the Fig.3.8.1. Let the height of the cylinder be Δh and be placed in such a way that $\Delta h/2$ is in medium 1 and remaining $\Delta h/2$ is in medium 2. Also the axis of the cylinder is in the normal direction to the surface.

According to the Gauss's law for the magnetic field,

$$\oint_S \vec{B}.d\vec{S} = 0 \ldots\ldots\ldots (1)$$

The surface integral must be evaluated over three surfaces, (i) Top, (ii) Bottom and (iii) Side.

Let the area of the top and bottom is same, equal to ΔS.

$$\therefore \qquad \oint_{top} \vec{B}.d\vec{S} + \oint_{bottom} \vec{B}.d\vec{S} + \oint_{side} \vec{B}.d\vec{S} = 0 \ldots\ldots (2)$$

As we are very much interested in the boundary conditions, reduce Δh to zero. As $\Delta h \to 0$, the cylinder tends to boundary and only top and bottom surfaces contribute in the surface integral. Thus surface integrals are calculated for top and bottom surfaces only. These surfaces are very small. Let the magnitude of normal component of $\vec{B}$ be $\vec{B}_{N1}$ and $\vec{B}_{N2}$ in medium 1 and medium 2 respectively. As both the surfaces are very small, we can assume $\vec{B}_{N1}$ and $\vec{B}_{N2}$ constant over their surfaces. Hence we can write,

For top surfaces:

$$\oint_{top} \vec{B}.d\vec{S} = \oint_{top} \vec{B}_{N1}.d\vec{S} = \left|\vec{B}_{N1}\right|\Delta S \ldots.. (3)$$

For bottom surface:

$$\oint_{bottom} \vec{B}.d\vec{S} = \oint_{bottom} \vec{B}_{N2}.d\vec{S} = -\left|\vec{B}_{N2}\right|\Delta S \ldots\ldots\ldots.. (4)$$

For side surface

$$\oint_{side} \vec{B}.d\vec{S} = 0 \ \ldots\ldots\ldots (5)$$

Putting values of surface integrals in equation (2), we get

$$\left|\vec{B}_{N1}\right|\Delta S - \left|\vec{B}_{N2}\right|\Delta S = 0$$

$$\therefore \qquad \left|\vec{B}_{N1}\right| = \left|\vec{B}_{N2}\right| \ \ldots\ldots\ldots (6)$$

Note that the negative sign is used for one of the surface integrals because normal component of surface in medium 2 is opposite to the normal component of surface in medium 1.

Thus the normal component of B is continuous at the boundary. As the magnetic flux density and the magnetic field intensity are related by

$$\vec{B} = \mu\vec{H}$$

Thus. equation (6) can be written as,

$$\mu_1\vec{H}_{N1} = \mu_2\vec{H}_{N2}$$

$$\therefore \qquad \frac{\vec{H}_{N1}}{\vec{H}_{N2}} = \frac{\mu_1}{\mu_2} = \frac{\mu_0\mu_{r1}}{\mu_0\mu_{r2}} = \frac{\mu_{r1}}{\mu_{r2}} \ldots\ldots.. (7)$$

Hence the normal component of $\vec{H}$ is not continuous at the boundary. The field strengths in two media are inversely proportional to their relative permeabilities.

3.8.2 Boundary Conditions for Tangential Component

According to Ampere's circuital law,

$$\oint_L \vec{H}.d\vec{L} = I \ \ldots\ldots\ldots (9)$$

Consider a rectangular closed path abcda as shown in the Fig. 3.8.1. It is traced in clockwise direction as a-b-c-d-a. This closed path is placed in a plane normal to the boundary surface. Hence $\oint_L \vec{H}.d\vec{L}$ can be divided into 6 parts.

$$\oint_L \vec{H}.d\vec{L} = I = \int_a^b \vec{H}.d\vec{L} + \int_b^2 \vec{H}.d\vec{L} + \int_2^c \vec{H}.d\vec{L} + \int_c^d \vec{H}.d\vec{L} + \int_d^1 \vec{H}.d\vec{L} + \int_1^a \vec{H}.d\vec{L} \;\ldots\ldots\ldots\; (10)$$

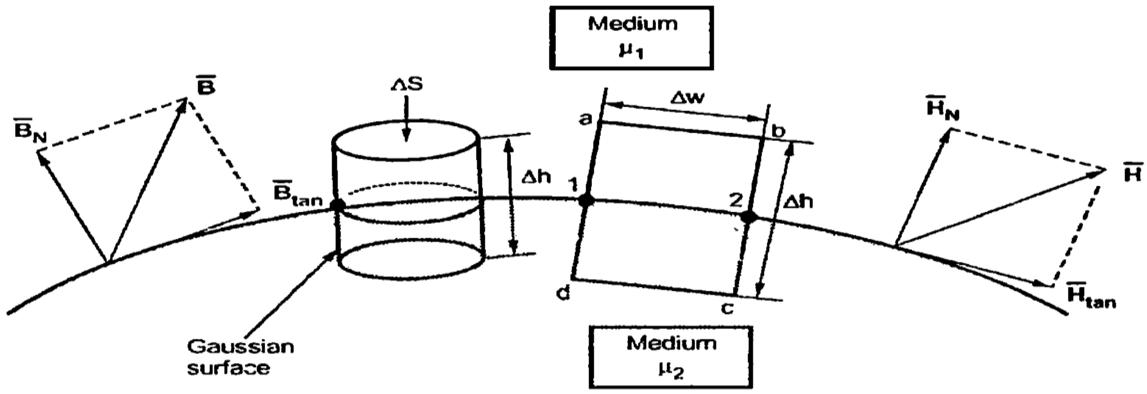


Fig. 3.8.1 Boundary between two magnetic materials of different permeabilities

From the Fig. 3.8.1 it is clear that, the closed path is placed in such a way that its two sides a-b and c-d are parallel to the tangential direction to the surface while the other two sides are normal to the surface at the boundary. This closed path is placed in such a way that half of its portion is in medium 1 and the remaining is in medium 2. The rectangular path is an elementary rectangular path with elementary height Δh and elementary width Δw. Thus over small width Δw, $\vec{H}$ can be assume constant say $\vec{H}_{tan1}$ in medium 1 and $\vec{H}_{tan2}$ in medium 2. Similarly, over a small height $\Delta h/2$, $\vec{H}$ can be assumed constant say $\vec{H}_{N1}$ in medium 1 and $\vec{H}_{N2}$ in medium 2. Now assume that $\vec{K}$ is the surface current. Thus equation (10) can be written as,

$$I = |\vec{K}|\Delta w = |\vec{H}_{tan1}|\Delta w + |\vec{H}_{N1}|\frac{\Delta h}{2} + |\vec{H}_{N2}|\frac{\Delta h}{2} - |\vec{H}_{tan2}|\Delta w - |\vec{H}_{N2}|\frac{\Delta h}{2} - |\vec{H}_{N1}|\frac{\Delta h}{2} \;\ldots\ldots\; (11)$$

To get conditions at boundary, $\Delta h \rightarrow 0$. Thus,

$$|\vec{K}|\Delta w = |\vec{H}_{tan1}|\Delta w - |\vec{H}_{tan2}|\Delta w$$

$$|\vec{K}| = |\vec{H}_{tan1}| - |\vec{H}_{tan2}| \ldots\ldots.. \; (12)$$

In vector form, we can express above relation by a cross product as

$$\vec{H}_{tan1} - \vec{H}_{tan2} = \vec{a}_{N12} \times \vec{K} \ldots\ldots\ldots (13)$$

Where $\vec{a}_{N12}$ is the unit vector in the direction normal at the boundary from medium 1to medium 2.

For $\vec{B}$ the tangential components can be related with permeabilitis of two media using equation (12),

$$|\vec{K}| = \frac{|\vec{B}_{tan1}|}{\mu_1} - \frac{|\vec{B}_{tan2}|}{\mu_2} \ldots\ldots (14)$$

Consider a special case that the boundary is free of current. In other words, media are not conductors; so K = 0. Then equation (12) becomes

$$\left|\vec{H}_{tan1}\right| - \left|\vec{H}_{tan2}\right| = 0 \ldots\ldots (15)$$

For tangential components of $\vec{B}$ we can write,

$$\frac{\left|\vec{B}_{tan1}\right|}{\mu_1} - \frac{\left|\vec{B}_{tan2}\right|}{\mu_2} = 0$$

$$\therefore \qquad \frac{\left|\vec{B}_{tan1}\right|}{\left|\vec{B}_{tan2}\right|} = \frac{\mu_1}{\mu_2} = \frac{\mu_{r1}}{\mu_{r2}} \ldots\ldots\ldots\ldots (16)$$

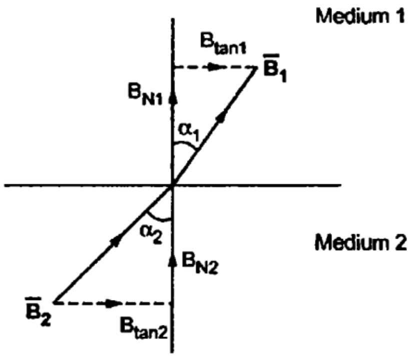


Fig. 3.8.2 Component of B̄ at boundary

Let the fields make angles α_1 and α_2 with the normal to the interface as shown in the Fig.3.8.2. In-terms of angle α_1 and α_2, we can write relationship between normal components of $\vec{B}$.

In medium 1,

$$\tan\alpha_1 = \frac{B_{tan1}}{B_{N1}} \ldots\ldots (17)$$

Similarly, in medium 2,

$$\tan\alpha_2 = \frac{B_{tan2}}{B_{N2}} \ldots\ldots (18)$$

Dividing equation (17) by equation (18)

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{B_{tan1}}{B_{N1}} \times \frac{B_{N2}}{B_{tan2}}$$

As we know, $B_{N1} = B_{N2}$,

$$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{B_{tan1}}{B_{tan2}} = \frac{\mu_{r1}}{\mu_{r2}}$$

Example 8.7 : *In region 1, as shown in the Fig. 8.16.*

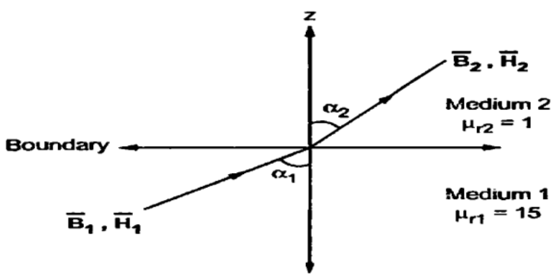


Fig. 8.16

$$\bar{B}_1 = 1.2\bar{a}_x + 0.8\bar{a}_y + 0.4\bar{a}_z \ T$$

Determine $\bar{B}_2$ and $\bar{H}_2$ in other medium and also calculate the angles made by the fields with the normal.

Solution : Assume that the boundary is current free.

In medium 1,

$$\bar{H}_1 = \frac{\bar{B}_1}{\mu_1} = \frac{\bar{B}_1}{\mu_0\mu_{r1}} = \frac{1}{\mu_0}\left[\frac{1.2\bar{a}_x + 0.8\bar{a}_y + 0.4\bar{a}_z}{15}\right]$$

$$\therefore \quad \bar{H}_1 = \frac{1}{\mu_0}\left[0.08\bar{a}_x + 0.0533\bar{a}_y + 0.0266\bar{a}_z\right] \text{ A/m} \quad \ldots (1)$$

As per the boundary shown in the Fig. 8.16, x and y are tangential, while z component is normal. So according to the boundary conditions for current free boundary, the tangential component for $\bar{H}_1$ and $\bar{H}_2$ remain same. The normal component can be calculated as

$$\frac{H_{N1}}{H_{N2}} = \frac{\mu_{r2}}{\mu_{r1}}$$

$$\therefore \quad H_{N2} = \frac{\mu_{r1}}{\mu_{r2}} H_{N1}$$

From equation (1), the normal component i.e. component in z-direction is

$$H_{N1} = 0.0266$$

$$\therefore \quad H_{N1} = \frac{15}{1}(0.0266) = 0.399 = 0.4 \text{ A/m} \quad \ldots (2)$$

Hence in medium 2,

$$H_{tan2\,x} = H_{tan1\,x} = 0.08,$$

$$H_{tan2\,y} = H_{tan1\,y} = 0.0533,$$

$$H_{N2} = 0.4$$

$$\therefore \quad \bar{H}_2 = \frac{1}{\mu_0}\left[0.08\,\bar{a}_x + 0.0533\,\bar{a}_y + 0.4\,\bar{a}_z\right] \text{ A/m} \quad \ldots (3)$$

Then the magnetic flux density in medium 2 is given by

$$\bar{B}_2 = \mu_2\,\bar{H}_2 = (\mu_0\mu_{r2})\bar{H}_2$$

$$\bar{B}_2 = \mu_0\left[\frac{1}{\mu_0}\left(0.08\bar{a}_x + 0.0533\bar{a}_y + 0.4\bar{a}_z\right)\right]$$

$$\therefore \quad \bar{B}_2 = 0.08\,\bar{a}_x + 0.0533\,\bar{a}_y + 0.4\,\bar{a}_z \text{ T} \quad \ldots (4)$$

As z-direction is perpendicular to boundary, in medium 1, we can write,

$$\bar{B}_1 \cdot \bar{a}_z = \left|\bar{B}_1\right|\left|\bar{a}_z\right|\cos\alpha_1$$

$$\therefore \left(1.2\bar{a}_x + 0.8\bar{a}_y + 0.4\bar{a}_z\right)\cdot\bar{a}_z = \sqrt{(1.2)^2 + (0.8)^2 + (0.4)^2}\,(1)(\cos\alpha_1)$$

$$\therefore \quad 0.4 = (1.4966)\ (\cos\alpha_1) \quad \ldots \bar{a}_x \cdot \bar{a}_x = \bar{a}_y \cdot \bar{a}_z = 0$$

$$\therefore \quad \alpha_1 \doteq 74.49^\circ$$

Similarly in medium 2,

$$\therefore \quad \cos\alpha_1 = \frac{24}{(49.6814)} = 0.483$$

$$\therefore \quad \alpha_1 = 61.12^\circ$$

Thus the angle made by B_1 with tangent to the interface is given by,

$$\theta_1 = 90^\circ - \alpha_1 = 90^\circ - 61.12^\circ = 28.88^\circ$$

The angle made by $\bar{B}_2$ with normal i.e. $\bar{a}_z$ is given by,

$$\bar{B}_2 \cdot \bar{a}_z = \left|\bar{B}_2\right|\left|\bar{a}_z\right|\cos\alpha_2$$

$$\therefore (22\bar{a}_x + 24\bar{a}_z)\cdot\bar{a}_z = \sqrt{(22)^2 + (24)^2}\,(1)\cos\alpha_2$$

$$\therefore \quad \cos\alpha_2 = \frac{24}{(32.5576)} = 0.7371$$

$$\therefore \quad \alpha_2 = 42.51^\circ$$

Thus the angle made by $\bar{B}_2$ with tangent to the interface is given by

$$\theta_2 = 90^\circ - \alpha_2 = 90^\circ - 42.51^\circ = 47.49^\circ$$

$$\therefore \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\tan(28.88^\circ)}{\tan(47.49^\circ)} = \frac{0.5515}{1.0909} = 0.5055$$