ELECTRO MAGNETIC FIELDS (3-0-0)
LECTURE NOTES
B. TECH
(II YEAR – III SEM)

Prepared by:
SUDIPTA MOHANTY., Assistant Professor
Department of Electrical Engineering

Odisha University of Technology and Research, Bhubaneswar

Techno Campus, P.O.-Mahalaxmi Vihar
Bhubaneswar-751029, Odisha, India

Electromagnetic Fields (3-0-0)

Prerequisites:

1. Mathematics-I
2. Mathematics-II

Course Outcomes

At the end of the course, students will demonstrate the ability

1. To understand the basic laws of electromagnetism.
2. To obtain the electric and magnetic fields for simple configurations under static conditions.
3. To analyse time-varying electric and magnetic fields.
4. To understand Maxwell's equation in different forms and different media.
5. To understand the propagation of EM waves.

Module 1: (08 Hours)

Co-ordinate systems & Transformation: Cartesian co-ordinates, circular cylindrical coordinates, spherical coordinates. Vector Calculus: Differential length, Area & Volume, Line, surface and volume Integrals, Del operator, Gradient of a scalar, Divergence of a vector & Divergence theorem, Curl of a vector & Stoke's theorem, Laplacian of a scalar.

Module 2: (10 Hours)

Electrostatic Fields: Coulomb's Law, Electric Field Intensity, Electric Fields due to a point, line, surface and volume charge, Electric Flux Density, Gauss's Law- Maxwell's Equation, Applications of Gauss's Law, Electric Potential, Relationship between E and V- Maxwell's Equation and Electric Dipole & Flux Lines, Energy Density in Electrostatic Fields., Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions. Electrostatic boundary-value problems: Poisson's and Laplace's Equations, Uniqueness Theorem, General procedures for solving Poisson's and Laplace's equations, Capacitance.

Module 3: (06 Hours)

Magneto static Fields: Magnetic Field Intensity, Biot-Savart's Law, Ampere's circuit Law-Maxwell Equation, applications of Ampere's law, Magnetic Flux Density-Maxwell's equations. Maxwell's equation for static fields, Magnetic Scalar and Vector potentials. Magnetic Boundary Conditions.

Module 4: (10 Hours)

Electromagnetic Field and Wave propagation: Faraday's Law, Transformer & Motional Electromagnetic Forces, Displacement Current, Maxwell's Equation in Final forms, Time-Harmonic Field. Electromagnetic Wave Propagation: Wave Propagation in lossy Dielectrics, Plane Waves in loss less Dielectrics, Free space, Good conductors Power & Poynting vector.

TEXTBOOKS:

1. Matthew N. O. Sadiku, Principles of Electromagnetics, 6th Ed., Oxford Intl. Student Edition, 2014.

REFERENCE BOOKS:

1. C. R. Paul, K. W. Whites, S. A. Nasor, Introduction to Electromagnetic Fields, 3rd Ed, TMH.
2. W.H. Hyat, Electromagnetic Field Theory, 7th Ed, TMH.
3. A. Pramanik, "Electromagnetism - Theory and applications", PHI Learning Pvt. Ltd, New Delhi, 2009.
4. A. Pramanik, "Electromagnetism-Problems with solution", Prentice Hall India, 2012.
5. G.W. Carter, "The electromagnetic field in its engineering aspects", Longmans, 1954.
6. W.J. Duffin, "Electricity and Magnetism", McGraw Hill Publication, 1980.
7. W.J. Duffin, "Advanced Electricity and Magnetism", McGraw Hill, 1968.
8. E.G. Cullwick, "The Fundamentals of Electromagnetism", Cambridge University Press, 1966.
9. B. D. Popovic, "Introductory Engineering Electromagnetics", Addison- Wesley Educational Publishers, International Edition, 1971.
10. W. Hayt, "Engineering Electromagnetics", McGraw Hill Education, 2012.

MODULE-II

1. Electrostatic Field
   - Coulomb’s law
   - Electric Field Intensity
   - Electric Field due to a point, line, surface and volume charge
   - Electric Flux Density
   - Gauss’s Law-Maxwell’s Equation
   - Application of Gauss’s law
   - Electric Potential
   - Relationship between E and V- Maxwell's Equation
   - Electric Dipole & Flux Lines
   - Energy Density in Electrostatic Fields
   - Current and current density
   - Ohms Law in Point form
   - Continuity of current
   - Boundary conditions.

2. Electrostatic boundary-value problems
   - Poisson's and Laplace's Equations
   - Uniqueness Theorem
   - General procedures for solving Poisson's and Laplace's equations
   - Capacitance

Chapter-2

Electrostatic Field

2.1 Introduction

The study of time invariant electric field in space or vacuum, produced by various types of static charge distributions is called **electrostatic field**. A very common example of such a field used in cathode ray tube for focusing and deflecting a beam. Most of the computer peripheral devices like keyboards, touch pads, liquid crystal displays etc. work on the principle of electrostatics. A variety of machines such as X-ray machine and medical instruments used for electrocardiograms, scanning etc. use the principle of electrostatics. Many industrial processes like spray painting, electrodeposition etc. also use the principle of electrostatics. Electrostatics also used in the agricultural activities like strong seeds, spraying to plants etc. many components such as resistors, capacitors etc. and the devices such as bipolar transistors field effect transistors function based on electrostatics.

2.2 Coulomb's Law

Key Point: A point charge means charges that are located on a body whose dimensions are much smaller than other relevant dimensions e.g. a collection of electric charges on a pin head may be regarded as a point charge.

The Coulomb's law states that force between the two point charges Q_1 and Q_2,

1. Acts along the line joining the two point charges.
2. Is directly proportional to the product (Q_1Q_2) of the two charges.
3. Is inversely proportional to the square of the distance between them.

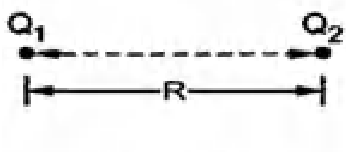

Fig. 2.2.1

Consider two point charges Q_1 and Q_2 as shown in Fig. 2.2.1, separated by the distance R. the charge Q_1 exerts a force on Q_2 while Q_2 also exerts a force on Q_1. The force acts along the line joining Q_1 and Q_2. The force exerted between them is repulsive if the charges are of same polarity while it is attractive if the charges are of different polarity.

Mathematically the force F between the charges can be expressed as,

$$F \propto \frac{Q_1 Q_2}{R^2} \ldots\ldots.. (1)$$

Where Q_1Q_2 = product of two charges

R = distance between two charges

The Coulomb's law also states that this force depends on the medium in which the point charges are located. The effect of medium is introduced in the equation of force as a constant of proportionality denoted as k.

Hence $$F = k\frac{Q_1Q_2}{R^2} \quad \text{.....(2)}$$

Where k = constant of proportionality

In International System of Units (SI), the charges Q_1 and Q_2 are expressed in Coulombs (C), the distance R in meters (m) and the force F in newtons (N). Then to satisfy Coulomb's law, the constant of proportionality is defined as,

$$k = \frac{1}{4\pi\varepsilon} \quad \text{....... (3)}$$

Where ε = Permittivity of the medium in which charges are located.

The units of ε are farads/meter(F/m).

In general, ε is expressed as,

$$\varepsilon = \varepsilon_0\varepsilon_r \quad \text{...... (4)}$$

where ε_0 = permittivity of free space or vacuum

ε_r = Relative permittivity or dielectric constant of the medium with respect to free space

ε = Absolute permittivity

For the free space or vacuum, the relative permittivity ε_r =1, hence $\varepsilon = \varepsilon_0$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2} \quad \text{...... (5)}$$

The value of permittivity of free space ε_0 is,

$$\varepsilon_0 = \frac{1}{36\pi} \times 10^{-9} = 8.854 \times 10^{-12}\,\text{F/m} \quad \text{...... (6)}$$

$$\therefore k = \frac{1}{4\pi\varepsilon_0} = \frac{1}{4\pi \times 8.854 \times 10^{-12}} = 8.98 \times 10^9 = 9 \times 10^9\,\text{m/F} \quad \text{......... (7)}$$

Hence the Coulomb's law can be expressed as,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2} \quad \text{....... (8)}$$

This is the force between the two point charges located in free space or vacuum.

2.2.1 Vector Form of Coulomb's Law

The force exerted between the two point charges has a fixed direction which is a straight line joining the two charges. Hence the force exerted between the two charges can be expressed in a vector form.

Consider the two point charges Q_1 and Q_2 located at the points having position vectors $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ as shown in Fig. 2.2.2.

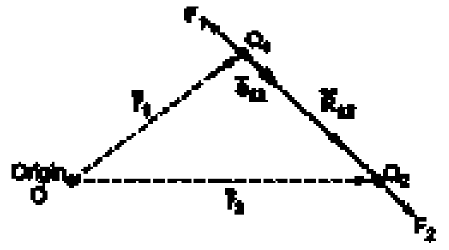

Fig.2.2.2 Vector form of Coulomb's law

Then the force exerted by Q_1 and Q_2 acts along the direction $\overrightarrow{R_{12}}$ where $\overrightarrow{a_{12}}$ is unit vector along $\overrightarrow{R_{12}}$. Hence the force in the vector form can be expressed as,

$$\overrightarrow{F_2} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{{R_{12}}^2}\overrightarrow{a_{12}} \quad \ldots\ldots. (9)$$

Where $\quad \overrightarrow{a_{12}}$ = Unit vector along $\overrightarrow{R_{12}} = \frac{\text{vector}}{\text{Magnitude of vector}}$

$$\therefore \quad \overrightarrow{a_{12}} = \frac{\overrightarrow{R_{12}}}{|\overrightarrow{R_{12}}|} = \frac{\overrightarrow{r_2}-\overrightarrow{r_1}}{|\overrightarrow{R_{12}}|} = \frac{\overrightarrow{r_2}-\overrightarrow{r_1}}{|\overrightarrow{r_2}-\overrightarrow{r_1}|} \quad \ldots\ldots.. (10)$$

Where $\quad \left|\overrightarrow{R_{12}}\right|$ = R = distance between the two charges

The following observations are important:

1. As shown in the Fig. 2.2.2, the force F_1 is the force exerted on Q1 due to Q2. It can be expressed as,

$$\overrightarrow{F_1} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{{R_{21}}^2}\overrightarrow{a_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^2} \times \frac{\overrightarrow{r_1}-\overrightarrow{r_2}}{|\overrightarrow{r_1}-\overrightarrow{r_2}|}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{|\overrightarrow{r_1}-\overrightarrow{r_2}|^3} \times (\overrightarrow{r_1}-\overrightarrow{r_2}) \quad \ldots\ldots\ldots (11)$$

But $\quad \overrightarrow{r_2}-\overrightarrow{r_1} = -[\overrightarrow{r_1}-\overrightarrow{r_2}]$

$$\overrightarrow{a_{21}} = -\overrightarrow{a_{12}}$$

Hence substituting in (11),

$$\overrightarrow{F_1} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{R^2}(-\overrightarrow{a_{12}}) = -\overrightarrow{F_2} \ldots\ldots\ldots (12)$$

Hence force exerted by the two charges on each other is equal but opposite in direction.

2. The like charges repel each other while the unlike charges attract each other. This is shown in Fig. 2.2.3. These are experiment conclusions though not reflected in the mathematical expression.

Fig. 2.2.3

3. It is necessary that the two charges are the point charges and stationary in nature.
4. The two point charges may be positive or negative. Hence their signs must be considered while using equation (9) to calculate the force exerted.
5. The Coulomb's law is linear which shows that if any one charge is increased 'n' times then the force exerted also increases by n times.

$$\therefore \quad \overrightarrow{F_2} = -\overrightarrow{F_1} \text{ then } n\overrightarrow{F_2} = -n\overrightarrow{F_1}$$

Where n = scalar

2.2.2 Principle of Superposition Theorem

If there are more than two point charges, then each will exert force on the other, then the net force on any charge can be obtained by the **principle of superposition.**

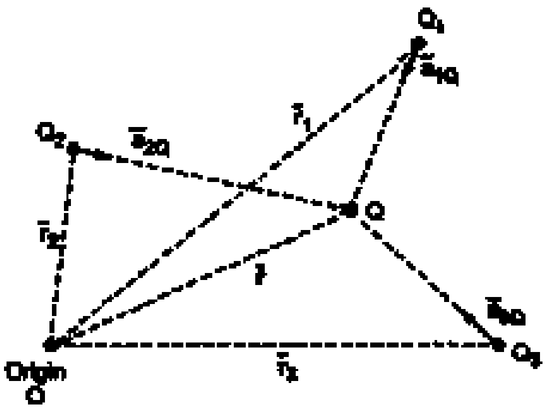

Fig. 2.2.4

If we have more than two point charges, we can use the principle of superposition to determine the force on a particular charge. The principle states that if there are 'n' charges Q_1, Q_2, ……, Qn located respectively, at points with position vectors $\overrightarrow{r_1}, \overrightarrow{r_2}, \ldots \ldots, \overrightarrow{r_n}$, the resultant force F on a charge Q located at point $\vec{r}$ is the vector sum of the forces exerted on Q by each of the charges Q_1, Q_2, ……, Qn.

Hence

$$\vec{F} = \frac{QQ_1}{4\pi\varepsilon_0}\frac{\vec{r}-\overrightarrow{r_1}}{|\vec{r}-\overrightarrow{r_1}|^3} + \frac{QQ_2}{4\pi\varepsilon_0}\frac{\vec{r}-\overrightarrow{r_2}}{|\vec{r}-\overrightarrow{r_2}|^3} + \cdots \ldots \ldots . . + \frac{QQ_n}{4\pi\varepsilon_0}\frac{\vec{r}-\overrightarrow{r_n}}{|\vec{r}-\overrightarrow{r_n}|^3} \ldots \ldots (13)$$

$$\therefore \ \vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{n} Q_k \frac{\vec{r}-\overrightarrow{r_k}}{|\vec{r}-\overrightarrow{r_k}|^3} \ \ldots\ldots\ldots \ (14)$$

Example 2.2 : *Four point charges each of 10 μC are placed in free space at the points (1, 0, 0), (−1, 0, 0), (0, 1, 0) and (0, − 1, 0) m respectively. Determine the force on a point charge of 30 μC located at a point (0, 0, 1) m.*

Solution : Use the principle of superposition as there are four charges exerting a force on the fifth charge. The locations of charges are shown in the Fig. 2.6.

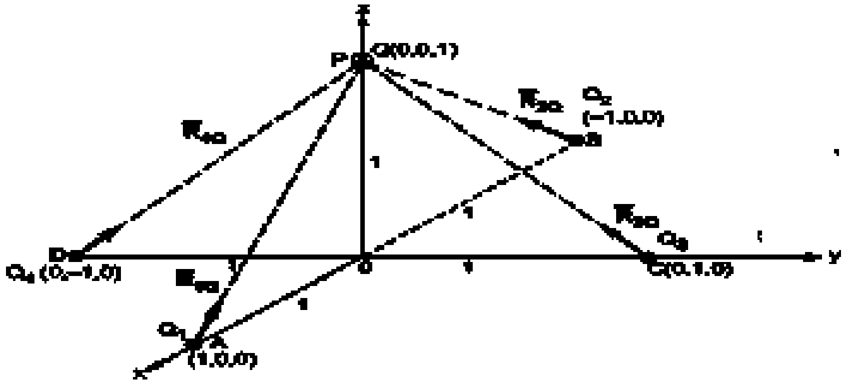

Fig. 2.6

The position vectors of four points at which the charges Q_1 to Q_4 are located can be obtained as,

$$\bar{A} = \bar{a}_x, \ \bar{B} = -\bar{a}_x, \ \bar{C} = \bar{a}_y \ \text{ and } \ \bar{D} = -\bar{a}_y$$

while position vector of point P where charge of 30 μC is situated is,

$$\bar{P} = \bar{a}_z$$

Consider force on Q due to Q_1 alone,

$$\bar{F}_1 = \frac{Q Q_1}{4\pi\varepsilon_0 R_{1Q}^2}\bar{a}_{1Q} = \frac{Q Q_1}{4\pi\varepsilon_0 R_{1Q}^2}\cdot\frac{\bar{R}_{1Q}}{\left|\bar{R}_{1Q}\right|}$$

where $\quad \bar{R}_{1Q} = \bar{P} - \bar{A} = \bar{a}_z - \bar{a}_x \quad$ and $\quad \left|\bar{R}_{1Q}\right| = \sqrt{1^2+1^2} = \sqrt{2}$

$$\therefore \quad \bar{F}_1 = \frac{30\times10^{-6}\times10\times10^{-6}}{4\pi\times8.854\times10^{-12}\times\left(\sqrt{2}\right)^2}\left[\frac{\bar{a}_z - \bar{a}_x}{\sqrt{2}}\right]$$

$$= 0.9533\,[\bar{a}_z - \bar{a}_x] \quad \text{... (1)}$$

It can be seen from the Fig. 2.6 that due to symmetry,

$$|\bar{R}_{1Q}| = |\bar{R}_{2Q}| = |\bar{R}_{3Q}| = |\bar{R}_{4Q}| = \sqrt{2}$$

Now

$$\bar{R}_{2Q} = \bar{P} - \bar{B} = \bar{a}_y + \bar{a}_x, \quad \bar{a}_{2Q} = \bar{a}_y + \bar{a}_x/\sqrt{2}$$

$$\bar{R}_{3Q} = \bar{P} - \bar{C} = \bar{a}_x - \bar{a}_y, \quad \bar{a}_{3Q} = \bar{a}_x - \bar{a}_y/\sqrt{2}$$

$$\bar{R}_{4Q} = \bar{P} - \bar{D} = \bar{a}_x + \bar{a}_y, \quad \bar{a}_{4Q} = \bar{a}_x + \bar{a}_y/\sqrt{2}$$

$$\therefore \quad \bar{F}_2 = \text{Force on Q due to } Q_2 = \frac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2}\bar{a}_{2Q}$$

$$\therefore \quad \bar{F}_3 = \text{Force on Q due to } Q_3 = \frac{QQ_3}{4\pi\varepsilon_0 R_{3Q}^2}\bar{a}_{3Q}$$

$$\therefore \quad \bar{F}_4 = \text{Force on Q due to } Q_4 = \frac{QQ_4}{4\pi\varepsilon_0 R_{4Q}^2}\bar{a}_{4Q}$$

$$\frac{QQ_2}{4\pi\varepsilon_0 R_{2Q}^2} = \frac{QQ_3}{4\pi\varepsilon_0 R_{3Q}^2} = \frac{QQ_4}{4\pi\varepsilon_0 R_{4Q}^2} = \frac{30\times10^{-6}\times10\times10^{-6}}{4\pi\times8.854\times10^{-12}\times(\sqrt{2})^2} = 1.3481$$

$$\therefore \quad \bar{F}_2 = 1.3481\left[\frac{\bar{a}_x + \bar{a}_x}{\sqrt{2}}\right] = 0.9533\,(\bar{a}_x + \bar{a}_x) \quad \text{... (2)}$$

$$\therefore \quad \bar{F}_3 = 1.3481\left[\frac{\bar{a}_x - \bar{a}_y}{\sqrt{2}}\right] = 0.9533\,(\bar{a}_x - \bar{a}_y) \quad \text{... (3)}$$

$$\therefore \quad \bar{F}_4 = 1.3481\left[\frac{\bar{a}_x + \bar{a}_y}{\sqrt{2}}\right] = 0.9533\,(\bar{a}_x + \bar{a}_y) \quad \text{... (4)}$$

Hence the total force $\bar{F}_t$ exerted on Q due to all four charges is vector sum of the individual forces exerted on Q by the charges.

$$\therefore \quad \bar{F}_t = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$$

$$= 0.9533\,[\bar{a}_x - \bar{a}_x + \bar{a}_x + \bar{a}_x + \bar{a}_x - \bar{a}_x + \bar{a}_x + \bar{a}_x] = 3.813\,\bar{a}_x \text{ N}$$

2.3 Electric Field Intensity

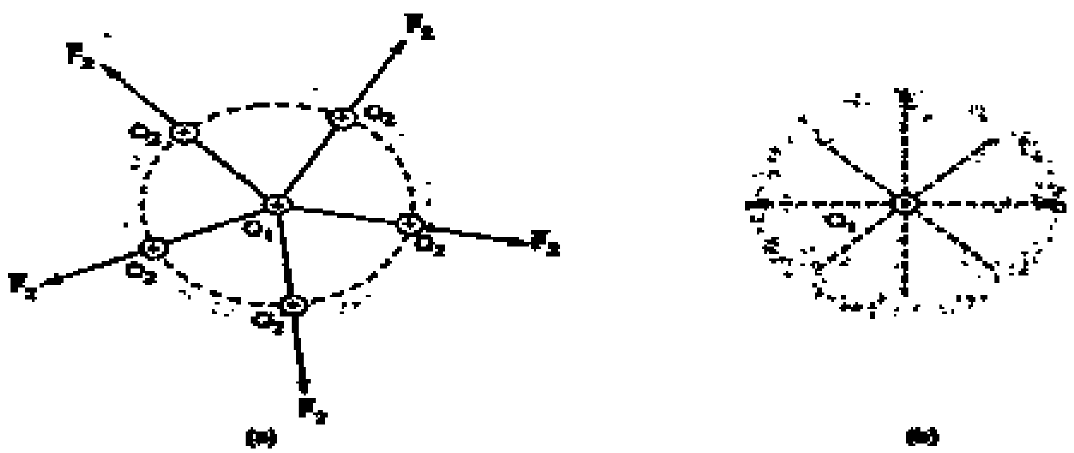

Fig 2.3.1 Electric field

Consider a point charge Q_1 as shown in Fig. 2.3.1. If any other similar charge Q_2 is brought near it, Q_2 experiences a force. In fact, if Q_2 is moved around Q_1, still Q_2 experiences a force as shown in Fig. 2.3.1. (a).

Thus there exists a region around a charge in which it exerts a force on any other charge. This region where a particular charge exerts a force on any other charge located in that region is called **electric field** of that charge. The electric field of Q_1 is shown in Fig. 2.3.1. (b).

The force experienced by the charge Q_2 due to Q_1 is given by Coulomb's law as,

$$\overrightarrow{F_2} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 {R_{12}}^2}\overrightarrow{a_{12}}$$

The force per unit charge can be written as,

$$\frac{\overrightarrow{F_2}}{Q_2} = \frac{Q_1}{4\pi\varepsilon_0 {R_{12}}^2}\overrightarrow{a_{12}} \text{.........} \ (1)$$

This force exerted per unit charge is called **electric field intensity or electric field strength.** It is a vector quantity and is directed along a segment from the charge Q_1 to the position of any other charge. It is denoted as $\vec{E}$.

$$\therefore \quad \vec{E} = \frac{Q_1}{4\pi\varepsilon_0 {R_{1p}}^2}\overrightarrow{a_{1p}} \text{.......} \ (2)$$

Where p = position of any other charge around Q_1.

2.3.1 Unit of $\vec{E}$

The definition of electric field intensity is,

$$\vec{E} = \frac{\text{Force}}{\text{Unit charge}} = \frac{\text{(N)Newtons}}{\text{(C)Coulomb}}$$

Hence units of $\vec{E}$ is N/C and it is also measured in units V/m (volts per meter).

2.3.2 Electric Field due to Discrete Charges

Consider n charges Q_1, Q_2, ……. Q_n as shown in the Fig. 2.3.2. The combined electric field intensity is to be obtained at point P. The distances of point P from Q_1, Q_2, ……. Q_n are R_1, R_2, ……. R_n respectively. The unit vectors along these directions are $\overrightarrow{a_{R1}}, \overrightarrow{a_{R2}} \dots \dots \dots \overrightarrow{a_{Rn}}$ respectively.

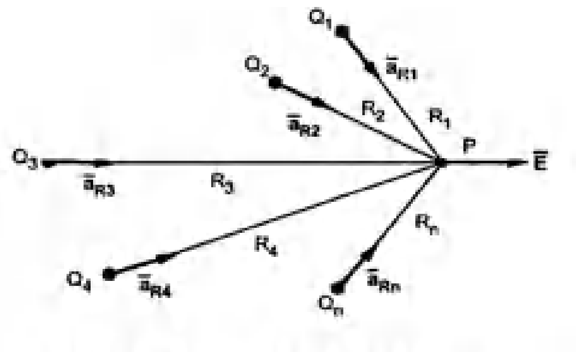

Fig. 2.3.2 $\bar{E}$ due to n number of charges

Then the total electric field intensity at point P is the vector sum of the individual field intensities produced by various charges at the point P.

$$\vec{E} = \overrightarrow{E_1} + \overrightarrow{E_2} + \overrightarrow{E_3} + \cdots \ldots\ldots + \overrightarrow{E_n}$$

$$= \frac{Q_1}{4\pi\varepsilon_0 {R_1}^2}\overrightarrow{a_{R1}} + \frac{Q_2}{4\pi\varepsilon_0 {R_2}^2}\overrightarrow{a_{R2}} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 {R_n}^2}\overrightarrow{a_{Rn}}$$

$\therefore$ $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{Q_i}{{R_i}^2}\overrightarrow{a_{Ri}} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Each unit vector can be obtained by using the method discussed earlier.

$$\overrightarrow{a_{Ri}} = \frac{\overrightarrow{r_p} - \overrightarrow{r_i}}{\left|\overrightarrow{r_p} - \overrightarrow{r_i}\right|}$$

Where $\overrightarrow{r_p}$ = Position vector of point P

$\overrightarrow{r_i}$ = Position vector of point where charge Qi is placed

➡ **Example 2.3 :** *Determine the electric field intensity at P(– 0.2, 0, – 2.3) m due to a point charge of +5 nC at Q(0.2, 0.1, – 2.5) m in air.*

Solution :

$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\bar{a}_R$$

$$\bar{a}_R = \frac{\bar{R}_{QP}}{|\bar{R}_{QP}|} = \frac{\bar{P}-\bar{Q}}{|\bar{P}-\bar{Q}|}$$

$$\bar{P}-\bar{Q} = (-0.2-0.2)\bar{a}_x + (0-0.1)\bar{a}_y + [-2.3-(-2.5)]\bar{a}_z$$

$$= -0.4\bar{a}_x - 0.1\bar{a}_y + 0.2\bar{a}_z$$

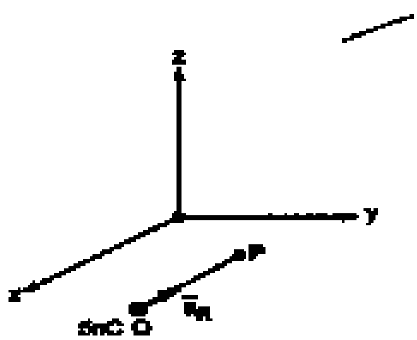

Fig. 2.11

$\therefore$ $$\bar{a}_R = \frac{-0.4\bar{a}_x - 0.1\bar{a}_y + 0.2\bar{a}_z}{\sqrt{(-0.4)^2 + (0.1)^2 + (0.2)^2}}$$

$$= \frac{-0.4\bar{a}_x - 0.1\bar{a}_y + 0.2\bar{a}_z}{0.45825}$$

$$= -0.8728\bar{a}_x - 0.2182\bar{a}_y + 0.4364\bar{a}_z$$

$\therefore \quad R = |\bar{P} - \bar{Q}| = 0.45825$

$$\therefore \quad \bar{E} = \frac{5\times10^{-9}}{4\pi\times 8.854\times10^{-12}\times(0.45825)^2}[\bar{a}_R] = 214\,\bar{a}_R$$

Substituting value of $\bar{a}_R$,

$$\bar{E} = -186.779\,\bar{a}_x - 46.694\,\bar{a}_y + 93.389\,\bar{a}_z \text{ V/m}$$

This is electric field intensity at point P.

➡ **Example 2.4 :** ***Calculate the field intensity at a point on a sphere of radius 3 m, if a positive charge of 2 μC is placed at the origin of the sphere.***

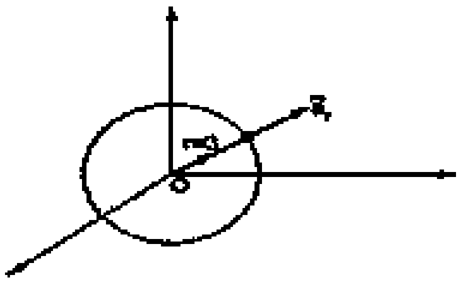

Fig. 2.12

Solution : Let us use spherical co-ordinate system.

The sphere of radius r = 3 m.

$\therefore \quad R = r = 3 \text{ m}$

And $\bar{E}$ acts radially outwards along the unit vector $\bar{a}_r$ in spherical co-ordinate system.

$\therefore \quad \bar{a}_R = \bar{a}_r$ in this case.

$$\therefore \quad \bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

$$\therefore \quad \bar{E} = \frac{2\times10^{-6}}{4\pi\times 8.854\times10^{-12}\times(3)^2}\bar{a}_r$$

$$= 1.9972\,\bar{a}_r \text{ kV/m}$$

Note that in this case $\bar{a}_r$ is specifically unit vector in spherical co-ordinate system, which is special case of general $\bar{a}_R$.

2.3.3 $\vec{E}$ due to Line Charge

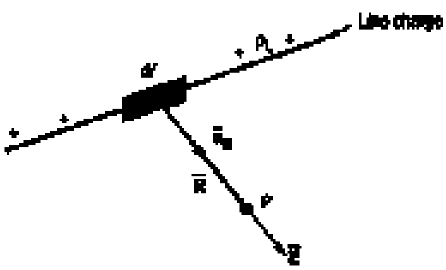

Fig. 2.3.3

Consider a line charge distribution having a charge density ρ_L as shown in Fig. 2.3.3.

The charge dQ on the differential length dl is, $dQ = \rho_L dl$.

Hence the differential electric field $\overrightarrow{dE}$ at point P due to dQ is given by,

$$\overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} = \frac{\rho L dl}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} \text{........ (4)}$$

Hence the total $\vec{E}$ at a point P due to line charge can be obtained by integrating $\overrightarrow{dE}$ over the length of the charge.

$$\vec{E} = \int_L \frac{\rho\text{Ldl}}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} \ldots\ldots (5)$$

The $\overrightarrow{a_R}$ and dl is to be obtained depending upon the co-ordinate system used.

2.3.4 $\vec{E}$ due to Surface Charge

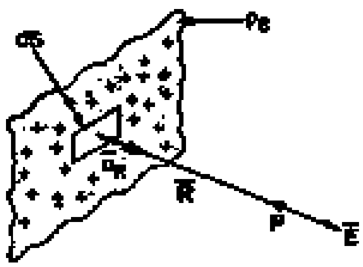

Fig. 2.3.4

Consider a surface charge distribution having a charge density ρ_S as shown in Fig. 2.3.4.

The charge dQ on the differential area dS is, $dQ = \rho_S dS$.

Hence the differential electric field $\overrightarrow{dE}$ at point P due to dQ is given by,

$$\overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} = \frac{\rho_S \text{dS}}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} \ldots\ldots\ldots (6)$$

Hence the total $\vec{E}$ at a point P due to surface charge can be obtained by integrating $\overrightarrow{dE}$ over the surface area on which charge is distributed. Note that this will be a double integration.

$$\vec{E} = \int_S \frac{\rho_S \text{dS}}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} \ldots\ldots (7)$$

The $\overrightarrow{a_R}$ and dS is to be obtained according to the position of the sheet of charge and the co-ordinate system used.

2.3.5 $\vec{E}$ due to Volume Charge

Fig. 2.3.5

Consider a volume charge distribution having a charge density ρ_v as shown in Fig. 2.3.5.

The charge dQ on the differential volume dv is, $dQ = \rho_v dv$.

Hence the differential electric field $\overrightarrow{dE}$ at point P due to dQ is given by,

$$\overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} = \frac{\rho_v \text{dv}}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} \ldots\ldots\ldots (8)$$

Hence the total $\vec{E}$ at a point P due to volume charge can be obtained by integrating $\overrightarrow{dE}$ over the volume in which charge is accumulated.

$$\vec{E} = \int_v \frac{\rho_v \mathrm{dv}}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R} \ldots\ldots (9)$$

The $\overrightarrow{a_R}$ and dv must be obtained according to the co-ordinate system used.

Thus if there are all possible types of charge distributions, then the total $\vec{E}$ at a point is the vector sum of individual electric field intensities produced by each of the charges at a point under consideration.

$$\vec{E}_{total} = \overrightarrow{E_P} + \overrightarrow{E_l} + \overrightarrow{E_S} + \overrightarrow{E_v} \ldots\ldots\ldots (10)$$

Where $\overrightarrow{E_P}$, $\overrightarrow{E_l}$, $\overrightarrow{E_S}$ and $\overrightarrow{E_v}$ are the field intensities due to point, line, surface and volume charge distributions respectively.

2.3.6 Electric Field due to Infinite Line Charge

Consider an infinitely long straight-line carrying uniform line charge having density ρ_L C/m. Let this line lies along the z-axis from -∞ to ∞ and hence called infinite line charge. Let point P is on y-axis at which electric field intensity is to be determined. The distance of point P from origin is r as shown in the Fig. 2.3.6.

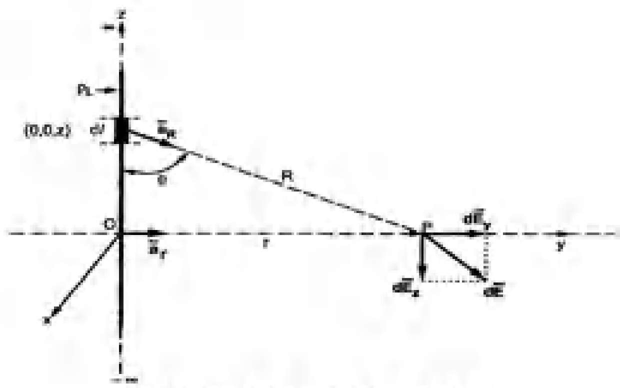

Fig. 2.3.6 Field due to infinite line charge

Consider a small differential length dl carrying a charge dQ, along the line as shown in the Fig. 2.3.6. It is along z-axis hence dl = dz.

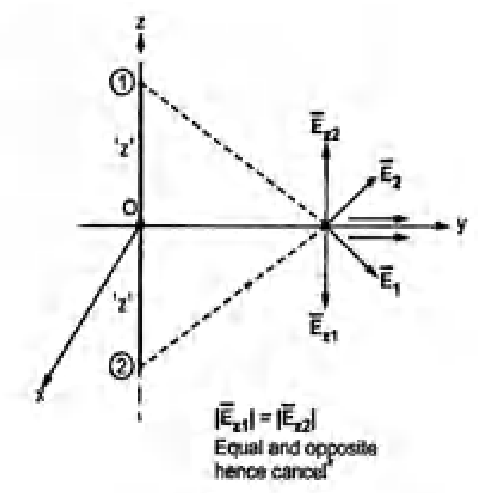

Fig. 2.3.7

$\therefore$ dQ = ρ_L dl = ρ_L dz……. (11)

The co-ordinates of dQ are (0, 0, z) while the co-ordinates of point p are (0, r, 0). Hence the distance vector $\vec{R}$ can be written as,

$$\vec{R} = \overrightarrow{r_P} - \overrightarrow{r_{dl}} = [r\overrightarrow{a_y} - z\overrightarrow{a_z}]$$

$$\therefore \quad |\vec{R}| = \sqrt{r^2 + z^2}$$

$$\therefore \quad \overrightarrow{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{[r\overrightarrow{a_y} - z\overrightarrow{a_z}]}{\sqrt{r^2+z^2}} \text{….. (12)}$$

$$\therefore \quad d\vec{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\overrightarrow{a_R}$$

$$= \frac{\rho_L dz}{4\pi\varepsilon_0(\sqrt{r^2+z^2})^2}\left[\frac{r\overrightarrow{a_y} - z\overrightarrow{a_z}}{\sqrt{r^2+z^2}}\right] \text{……… (13)}$$

Note: For every charge on positive z-axis, there is equal charge present on negative z-axis. Hence, the z-component of electric field intensities produced by such charges at point P will cancel each other. Hence, effectively there will not be any z component of $\vec{E}$ at P. This is shown in Fig. 2.3.7.

Hence the equation of $\overrightarrow{dE}$ can be written by eliminating $\overrightarrow{a_z}$ component,

$$\therefore \overrightarrow{dE} = \frac{\rho_L dz}{4\pi\varepsilon_0(\sqrt{r^2+z^2})^2}\left[\frac{r\overrightarrow{a_y}}{\sqrt{r^2+z^2}}\right] \text{…….. (14)}$$

Now by integrating $\overrightarrow{dE}$ over the z-axis from -∞ to ∞ we can obtain total $\vec{E}$ at point P.

$$\therefore \vec{E} = \int_{-\infty}^{\infty} \frac{\rho_L}{4\pi\varepsilon_0(r^2+z^2)^{3/2}} r dz\, \overrightarrow{a_y}$$

Note: For such an integration, use the substitution

$$z = r\tan\theta \text{ i.e. } r = \frac{z}{\tan\theta}$$

$$\therefore dz = r sec^2\theta\, d\theta$$

Here r is not the variable of integration.

For $z = -\infty, \theta = \tan^{-1}(-\infty) = -\frac{\pi}{2} = -90^0$

For $z = +\infty, \theta = \tan^{-1}(+\infty) = +\frac{\pi}{2} = +90^0$

$$\therefore \vec{E} = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho_L}{4\pi\varepsilon_0(r^2 + r^2 tan^2\theta)^{\frac{3}{2}}} r \times r\, sec^2\theta\, d\theta\overrightarrow{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2 sec^2\theta \, d\theta}{r^3(1 + tan^2\theta)^{\frac{3}{2}}} \overrightarrow{a_y}$$

But $\quad 1 + tan^2\theta = sec^2\theta$

$$\therefore \quad \vec{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{sec^2\theta \, d\theta}{r sec^3\theta} \overrightarrow{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos\theta \, d\theta \, \overrightarrow{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} [sin\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \overrightarrow{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} \left[sin\frac{\pi}{2} - \sin\left(\frac{-\pi}{2}\right) \right] \overrightarrow{a_y}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0 r} [1 - (-1)] \overrightarrow{a_y} = \frac{\rho_L}{4\pi\varepsilon_0 r} \times 2\overrightarrow{a_y}$$

$$\therefore \vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 r} \overrightarrow{a_y} \text{ V/m } \ldots\ldots\ldots (15)$$

The $\overrightarrow{a_y}$ is unit vector along the distance r which is perpendicular distance of point P from the line charge. Thus in general $\overrightarrow{a_y} = \overrightarrow{a_r}$.

Hence the result of $\vec{E}$ can be expressed as,

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 r} \overrightarrow{a_r} \text{ V/m} \ldots\ldots.. \; (16)$$

Where r = perpendicular distance of point from the line charge

$\overrightarrow{a_r}$ = unit vector in the direction of the perpendicular distance of point P from the line charge.

Very important notes:

1. The field intensity at any point has no component in the direction parallel to the line along which the charge is located and the charge is infinite.
2. The above equation consists of r and $\overrightarrow{a_r}$ which do not have meanings of cylindrical co-ordinate system. The distance r is to be obtained by distance formula while $\overrightarrow{a_r}$ is unit vector in the direction of $\overrightarrow{a_r}$.

Example 2.9 : *A uniform line charge, infinite in extent with ρ_L = 20 nC/m lies along the z axis. Find the $\bar{E}$ at (6,8,3) m.*

Solution : The line charge is shown in the Fig. 2.22.

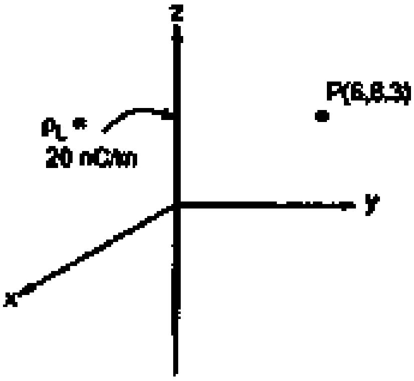

Fig. 2.22

Any point on the line is (0,0,z).

Key Point: *As line charge is along z axis, $\bar{E}$ can not have any component along z direction. So do not consider z co-ordinate while calculating $\bar{r}$.*

$$\therefore \quad \bar{r} = (6-0)\bar{a}_x + (8-0)\bar{a}_y$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{6\,\bar{a}_x + 8\,\bar{a}_y}{\sqrt{6^2+8^2}} = \frac{6\,\bar{a}_x + 8\,\bar{a}_y}{10}$$

$$= 0.6\,\bar{a}_x + 0.8\,\bar{a}_y$$

Thus, $\bar{E} = \dfrac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$

$$= \frac{20\times10^{-9}}{2\pi\times8.854\times10^{-12}\times10}[0.6\bar{a}_x + 0.8\,\bar{a}_y] = 10.7853\,\bar{a}_x + 14.38\,\bar{a}_y \text{ V/m}$$

Example 2.10 : *A uniform line charge ρ_L = 25 nC/m lies on the line $x=-3$ m and $y=4$ m in free space. Find the electric field intensity at a point (2, 3, 15) m.*

Solution : The line is shown in the Fig. 2.25. The line with x = – 3 constant and y = 4 constant is a line parallel to z axis as z can take any value. The $\bar{E}$ at P (2, 3, 15) is to be calculated.

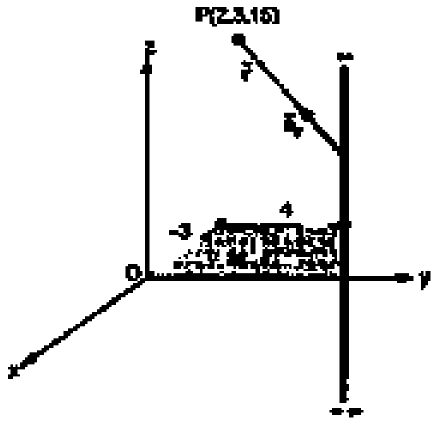

Fig. 2.25

The charge is infinite line charge hence $\bar{E}$ can be obtained by standard result.

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

To find $\bar{r}$, consider two points, one on the line which is (-3, 4, z) while P (2, 3, 15). But as line is parallel to z axis, $\bar{E}$ can not have component in $\bar{a}_z$ direction hence z need not be considered while calculating $\bar{r}$.

$$\therefore \quad \bar{r} = [2-(-3)]\bar{a}_x + [3-4]\bar{a}_y = 5\bar{a}_x - \bar{a}_y \qquad \text{... z not considered}$$

$$\therefore \quad |\bar{r}| = \sqrt{(5)^2+(-1)^2} = \sqrt{26}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{5\bar{a}_x - \bar{a}_y}{\sqrt{26}}$$

$$\therefore \quad \bar{E} = \frac{\rho_L}{2\pi\varepsilon_0}\cdot\frac{1}{\sqrt{26}}\left[\frac{5\bar{a}_x - \bar{a}_y}{\sqrt{26}}\right] = \frac{25\times10^{-9}[5\bar{a}_x - \bar{a}_y]}{2\pi\times8.854\times10^{-12}\times26}$$

$$= 86.42\,\bar{a}_x - 17.284\,\bar{a}_y \text{ V/m}$$

2.3.7 Electric Field due to Infinite Sheet of Charge

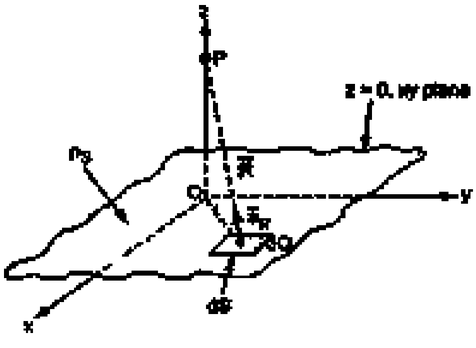

Fig. 2.3.7

Consider an infinite sheet of charge having uniform charge density ρ_s C/m^2, placed in xy-plane as shown in Fig. 2.3.7. Let us use cylindrical co-ordinates.

The point P at which $\vec{E}$ is to be calculated is on z-axis. Consider the differential surface area dS carrying charge dQ. The normal direction to dS is z direction hence dS normal to z direction is ρ dρ dϕ.

Now $$dQ = \rho_s dS = \rho_s\, \rho\, d\rho\, d\phi \text{ ………. (17)}$$

Hence, $$\overrightarrow{dE} = \frac{dQ}{4\pi\varepsilon_0 R^2}\vec{a}_R = \frac{\rho_s\, \rho\, d\rho\, d\phi}{4\pi\varepsilon_0 R^2}\vec{a}_R \text{ …….. (18)}$$

The distance vecor $\vec{R}$ has two components as shown in Fig. 2.3.8.

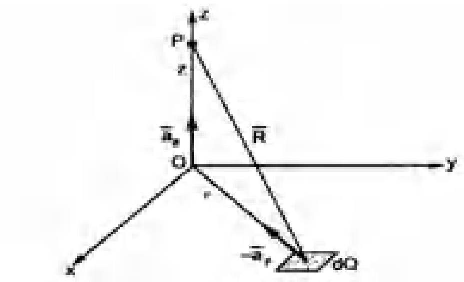

Fig. 2.3.8

1. The radial component ρ along -$\overrightarrow{a_\rho}$ i.e. -ρ$\overrightarrow{a_\rho}$.
2. The component z along $\overrightarrow{a_z}$ i.e. $z\overrightarrow{a_z}$.

With these two components $\vec{R}$ can be obtained from the differential area towards point P as,

$$\therefore \quad \vec{R} = -\rho\overrightarrow{a_\rho} + z\overrightarrow{a_z} \text{ …………………….. (19)}$$

$$\therefore \quad |\vec{R}| = \sqrt{(-\rho)^2 + (z)^2} = \sqrt{\rho^2 + z^2} \text{ …… (20)}$$

$$\therefore \quad \vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{-\rho\overrightarrow{a_\rho} + z\overrightarrow{a_z}}{\sqrt{\rho^2+z^2}} \text{ ………………….. (21)}$$

$$\therefore \quad \overrightarrow{dE} = \frac{\rho_s\, \rho\, d\rho\, d\phi}{4\pi\varepsilon_0 R^2}\left[\frac{-\rho\overrightarrow{a_\rho} + z\overrightarrow{a_z}}{\sqrt{\rho^2+z^2}}\right]$$

For infinite sheet in xy-plane, ρ varies from 0 to ∞ while ϕ varies from 0 to 2π.

Note: As there is symmetry about z-axis from all radial direction, all $\overrightarrow{a_\rho}$ component of $\vec{E}$ are going to cancel each other and net $\vec{E}$ will not have any radial component. Hence while integrating $\overrightarrow{dE}$ there is no need to consider $\overrightarrow{a_\rho}$ component.

$$\vec{E} = \int_{\phi=0}^{2\pi}\int_{0}^{\infty}\overrightarrow{dE} = \int_{0}^{2\pi}\int_{0}^{\infty}\frac{\rho_s\,\rho\,d\rho\,d\phi}{4\pi\varepsilon_0(\rho^2+z^2)^{3/2}}(z\vec{a}_z)$$

Put $\rho^2 + z^2 = u^2$ hence 2ρ dρ = 2u du

For ρ = 0, u = z and ρ = ∞, u = ∞, changing limits we have

$$\vec{E} = \int_{0}^{2\pi}\int_{u=z}^{\infty}\frac{\rho_S\,\text{u du}}{4\pi\varepsilon_0(u^2)^{3/2}}(z\vec{a}_z)$$

$$= \int_{0}^{2\pi}\int_{u=z}^{\infty}\frac{\rho_S\,\text{du}}{4\pi\varepsilon_0 u^2}(z\vec{a}_z)$$

$$= \int_{0}^{2\pi}\frac{\rho_S}{4\pi\varepsilon_0 u^2}d\phi(z\overrightarrow{a_z})\left[\frac{-1}{u}\right]_{z}^{\infty} \quad \text{as } \int\frac{1}{u^2} = \int u^{-2} = \frac{u^{-1}}{-1} = -\frac{1}{u}$$

$$= \frac{\rho_S}{4\pi\varepsilon_0}[\phi]_{0}^{2\pi}(z\vec{a}_z)\left[-\frac{1}{\infty} - \left(-\frac{1}{z}\right)\right] = \frac{\rho_S}{4\pi\varepsilon_0}(2\pi)\vec{a}_z$$

∴ $\vec{E} = \frac{\rho_S}{2\varepsilon_0}\vec{a}_z$V/m …..For points above xy plane

Now $\overrightarrow{a_z}$ is direction normal to differential surface area dS considered. Hence in general if $\overrightarrow{a_n}$ is direction normal to the surface containing charge, the above result can be generalized as,

∴ $\vec{E} = \frac{\rho_S}{2\varepsilon_0}\vec{a}_n$V/m ….. …. (22)

Where $\overrightarrow{a_n}$ = Direction normal to the surface charge

Thus for the points below xy plane ,$\vec{a}_n = -\vec{a}_z$ hence,

∴ $\vec{E} = -\frac{\rho_S}{2\varepsilon_0}\vec{a}_n$ V/m for points below xy plane

Key Points: Thus electric field due to infinite sheet of charge is every where normal to the surface and its magnitude is independent of the distance of a point from the plane containing the sheet of charge.

➡ Example 2.11 : *Charge lies in y = – 5m plane in the form of an infinite square sheet with a uniform charge density of ρ_s = 20 nC/m^2. Determine* E *at all the points.*

Solution : The plane y = – 5 m constant is parallel to xz plane as shown in the Fig. 2.29.

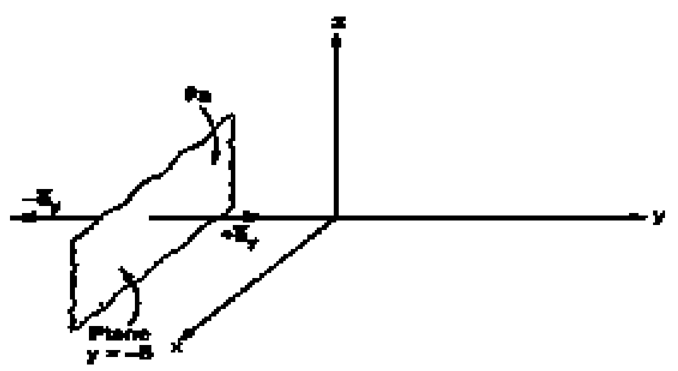

Fig. 2.29

For y > – 5, the E component will be along $+a_y$ as normal direction to the plane y = –5 m is a_y.

∴ $a_n = a_y$

$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}\,\bar{a}_n = \frac{\rho_S}{2\varepsilon_0}\,\bar{a}_y$

$$= \frac{20\times10^{-9}}{2\times8.854\times10^{-12}}\,\bar{a}_y = 1129.43\ \bar{a}_y \quad V/m$$

For $y < -5$, the $\bar{E}$ component will be along $-\bar{a}_y$ direction, with same magnitude.

$\therefore \quad \bar{E} = \frac{\rho_S}{2\varepsilon_0}\left(-\bar{a}_y\right) = -1129.43\ \bar{a}_y \quad V/m$

At any point to the left or right of the plane, $|\bar{E}|$ is constant and acts normal to the plane.

Example 2.12 : *Find $\bar{E}$ at P (1, 5, 2) m in free space if a point charge of 6 μC is located at (0,0,1), the uniform line charge density ρ_L = 180 nC/m along x axis and uniform sheet of charge with ρ_S = 25 nC/m^2 over the plane z = −1.*

Solution : **Case 1 :** Point charge Q_1 = 6 μC at A (0, 0, 1) and P (1, 5, 2)

$\therefore \quad \bar{E}_1 = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\,\bar{a}_{AP} = \frac{Q_1}{4\pi\varepsilon_0 R_{AP}^2}\left[\frac{\bar{R}_{AP}}{|\bar{R}_{AP}|}\right]$

$$\bar{R}_{AP} = (1-0)\bar{a}_x + (5-0)\bar{a}_y + (2-1)\bar{a}_z = \bar{a}_x + 5\bar{a}_y + \bar{a}_z$$

$\therefore \quad |\bar{R}_{AP}| = \sqrt{(1)^2+(5)^2+(1)^2} = \sqrt{27}$

$\therefore \quad \bar{E}_1 = \frac{6\times10^{-6}}{4\pi\times8.854\times10^{-12}\times\left(\sqrt{27}\right)^2}\left[\frac{\bar{a}_x + 5\bar{a}_y + \bar{a}_z}{\sqrt{27}}\right]$

$\therefore \quad \bar{E}_1 = 384.375\ \bar{a}_x + 1921.879\ \bar{a}_y + 384.375\ \bar{a}_z \quad V/m$

Case 2 : Line charge ρ_L along x axis.

It is infinite hence using standard result,

$$\bar{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r = \frac{\rho_L}{2\pi\varepsilon_0 r}\frac{\bar{r}}{|\bar{r}|}$$

Consider any point on line charge i.e. (x, 0, 0) while P (1, 5, 2). But as line is along x axis, no component of $\bar{E}$ will be along $\bar{a}_x$ direction. Hence while calculating $\bar{r}$ and $\bar{a}_r$, do not consider x co-ordinates of the points.

$\therefore \quad \bar{r} = (5-0)\bar{a}_y + (2-0)\bar{a}_z = 5\bar{a}_y + 2\bar{a}_z$

$\therefore \quad |\bar{r}| = \sqrt{(5)^2+(2)^2} = \sqrt{29}$

$\therefore \quad \bar{E}_2 = \frac{\rho_L}{2\pi\varepsilon_0\times\sqrt{29}}\left[\frac{5\bar{a}_y + 2\bar{a}_z}{\sqrt{29}}\right] = \frac{180\times10^{-9}\,[5\bar{a}_y + 2\bar{a}_z]}{2\pi\times8.854\times10^{-12}\times29}$

$$= 557.859\ \bar{a}_y + 223.144\ \bar{a}_z \quad V/m$$

Case 3 : Surface charge ρ_S over the plane z = – 1. The plane is parallel to xy plane and normal direction to the plane is $\bar{a}_n = \bar{a}_z$, as point P is above the plane. At all the points above z = – 1 plane the $\bar{E}$ is constant along $\bar{a}_z$ direction.

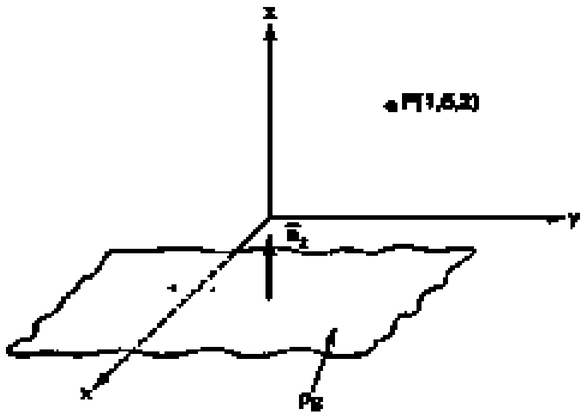

Fig. 2.35

$$\therefore \quad \bar{E}_3 = \frac{\rho_S}{2\varepsilon_0}\,\bar{a}_z$$

$$= \frac{25\times10^{-9}}{2\times8.854\times10^{-12}}\,\bar{a}_z$$

$$= 1411.7913\ \bar{a}_z\ \text{V/m}$$

Hence the net $\bar{E}$ at point P is,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 = 384.375\,\bar{a}_x + 1921.879\,\bar{a}_y + 384.375\,\bar{a}_z + 557.859\,\bar{a}_y + 223.144\,\bar{a}_z + 1411.7913\,\bar{a}_z$$

$$= \mathbf{384.375\,\bar{a}_x + 2479.738\,\bar{a}_y + 2019.3103\,\bar{a}_z\ \ V/m}$$

➡ **Example 2.17 :** *A circular ring of charge with radius 5 m lies in z = 0 plane with centre at origin. If the ρ_L = 10 nC/m, find the point charge Q placed at the origin which will produce same $\bar{E}$ at the point (0, 0, 5) m.*

Solution : The ring is shown in the Fig. 2.36 (a), in z = 0 i.e. xy plane.

The point P (0, 0, 5) m. Consider the differential length *dl* of the ring. It is in the φ direction hence dl = r dφ.

The charge on *dl* is $dQ = \rho_L\,dl$

$$\therefore \quad dQ = \rho_L\ r\,d\phi$$

$$\therefore \quad d\bar{E} = \frac{dQ}{4\pi\varepsilon_0 R^2}\,\bar{a}_R$$

$$= \frac{\rho_L\,r\,d\phi}{4\pi\varepsilon_0 R^2}\cdot\bar{a}_R$$

Now $\bar{a}_R = \frac{\bar{R}}{|\bar{R}|}$ and $\bar{R}$ can be resolved into two components as shown in the Fig. 2.36 (b).

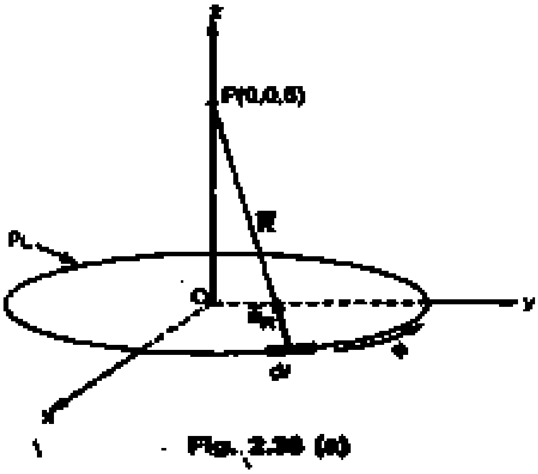

Fig. 2.36 (a)

The two components in cylindrical co-ordinate system are,

1. Along $-\bar{a}_r$ direction i.e. $-r\bar{a}_r$.

2. And z component in $\bar{a}_z$ direction i.e. $z\,\bar{a}_z$.

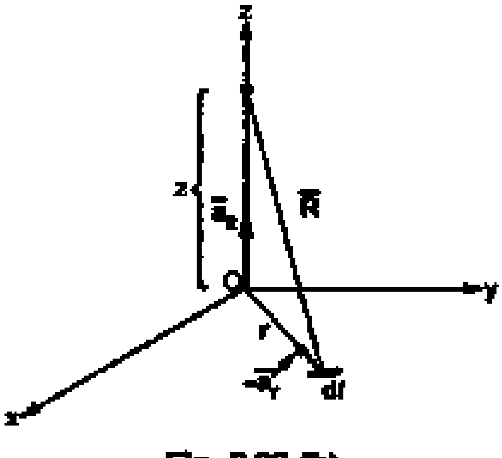

Fig. 2.36 (b)

$$\therefore \quad \bar{R} = -r\,\bar{a}_r + z\,\bar{a}_z$$

$$\text{hence} \quad |\bar{R}| = \sqrt{(r)^2 + z^2}$$

$$\therefore \quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2 + z^2}}$$

$$\therefore d\bar{E} = \frac{\rho_L \, r \, d\phi}{4\pi\varepsilon_0 \left(\sqrt{r^2+z^2}\right)^2} \left[\frac{-r\bar{a}_r + z\bar{a}_z}{\sqrt{r^2+z^2}}\right]$$

Note : The $\bar{E}$ at P will have two components, in radial direction and z direction but radial components are symmetrical about z axis, from all the points of the ring and hence will cancel each other. So there is no need to consider $\bar{a}_r$ component in integration. Though if considered, mathematically will get cancelled.

$$\therefore \quad \bar{E} = \int_{\phi=0}^{\phi=2\pi} \frac{\rho_L \, r \, d\phi}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}} z\bar{a}_z \qquad \text{... Limit for } \phi = 0 \text{ to } 2\pi$$

$$= \frac{\rho_L \, r z}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}} \left[\int_0^{2\pi} d\phi\right] \bar{a}_z \qquad \text{... } r = 5 \text{ m}, \; z = 5\text{m}$$

$$= \frac{\rho_L \, r z}{4\pi\varepsilon_0 \left(r^2+z^2\right)^{3/2}} (2\pi)\, \bar{a}_z = \frac{10\times10^{-9}\times5\times5\times2\pi}{4\pi\times8.854\times10^{-12}\times[25+25]^{3/2}} \bar{a}_z$$

$$\therefore \quad \bar{E} = 39.9314 \; \bar{a}_z \; \text{V/m} \qquad \text{... (1)}$$

Let Q be the point charge at the origin. From Q to point P, the distance vector $\bar{E} = 5\,\bar{a}_z$.

$$\therefore \bar{E} \text{ due to Q at P} = \frac{Q}{4\pi\varepsilon_0 R^2} \bar{a}_R$$

$$\text{where} \qquad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{5\bar{a}_z}{5} = \bar{a}_z$$

$$\therefore \bar{E} \text{ due to Q at P} = \frac{Q}{4\pi\varepsilon_0 (5)^2} \bar{a}_z \qquad \text{... (2)}$$

Equating (1) and (2),

$$\frac{Q}{4\pi\varepsilon_0\times25} = 39.9314$$

$$\therefore \quad Q = 111.071 \text{ nC}$$

2.4 Electric Flux

The total number of lines of force in any particular electric field is called **electric flux.** It is represented by the symbol Ψ. Similar to the charge, unit of electric flux is also coulomb C.

2.4.1 Properties of Electric Flux

1. The flux lines start from positive charge and terminate on the negative charge as shown in Fig.2.4.1.

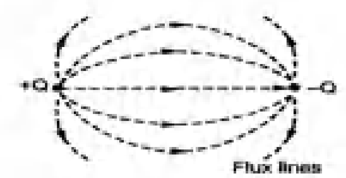

Fig.2.4.1Flux lines

2. If the negative charge is absent, then the flux lines terminates at infinity as shown in Fig. 2.4.2 (a). While in absence of positive charge, the electric flux terminates on the negative charge from infinity as shown in Fig. 2.4.2 (b).

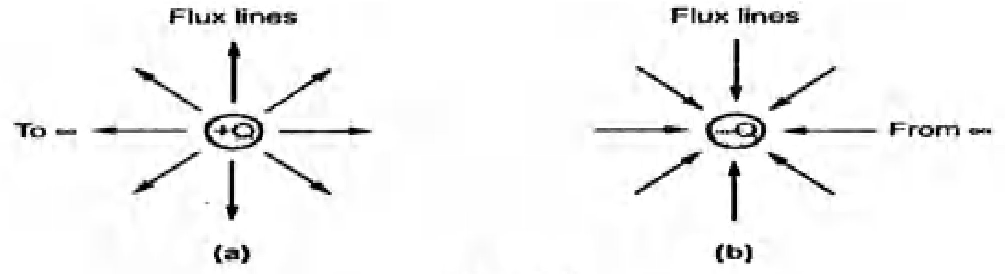

Fig.2.4.2

3. There are more number of lines i.e. crowding of lines if electric field is stronger.
4. These lines are parallel and never cross each other.
5. The lines are independent of the medium in which charges are placed.
6. The lines always enter or leave the charged surface, normally.
7. If the charge on a body is $\pm Q$ coulombs, then the total number of lines originating or terminating on it is also Q. But the total number of lines is nothing but a flux.

∴ Electric flux $\Psi = Q$ coulombs

The electric flux is also called **displacement flux**. The flux is a scalar field.

2.5 Electric Flux Density

Consider the two point charges as shown in Fig. 2.4.3. The flux lines originating from positive charge and terminating at negative charge. Consider a unit surface area as shown in Fig. 2.4.3. The number of flux lines are passing through this surface area.

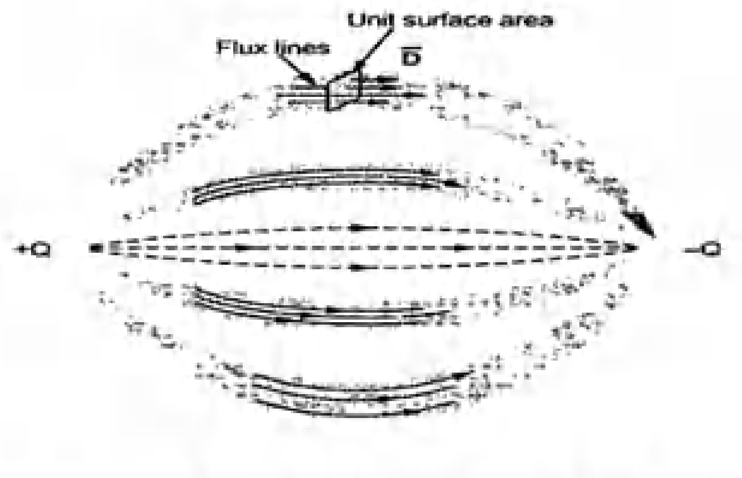

Fig.2.5.1 Concept of electric flux density

The net flux passing normal through the unit surface area is called the **electric flux density.** It is denoted as $\vec{D}$. It has a specific direction which is normal to the surface area under consideration hence it is a vector field.

Consider a sphere with a charge Q placed at its centre. There are no other charges present around. The total flux distributed radially around the charge is Ψ = Q. This flux uniformly distributes over the surface of the sphere. Then electric flux density is defined as,

$$D = \frac{\Psi}{S} \quad \text{in magnitude}$$

As Ψ is measured in coulombs and S in square meters, the units of D are C/m^2. This is also called **displacement flux density or displacement density**.

The flux density at any point can be represented in vector form as,

$$\vec{D} = \frac{\Psi}{S}\vec{a}_n \quad C/m^2 \ldots (1)$$

Where $\vec{a}_n$= unit vector in the direction normal to the surface area

2.5.1 $\vec{D}$ due to a point charge Q

Consider a point charge +Q placed at the centre of the imaginary sphere of radius r as shown in Fig. 2.5.2. The flux originated from the point charge +Q are directed radially outwards. The magnitude of the flux density at any point on the surface is,

$$|\vec{D}| = \frac{\text{Total flux } \Psi}{\text{Total surface area S}} \ldots\ldots\ldots (2)$$

But $\Psi = Q = \text{Total flux}$

And $S = 4\pi r^2 = \text{Total surface area}$

∴ $$|\vec{D}| = \frac{Q}{4\pi r^2} \ldots\ldots\ldots\ldots (3)$$

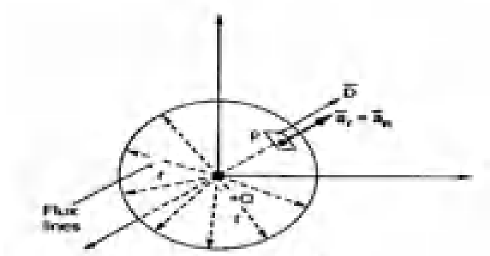

Fig. 2.5.2

The unit vector directed radially outwards and normal to the surface at any point on the sphere is $\vec{a_n} = \vec{a_r}$. Thus in the vector form, electric flux density at a point which is at a distance of r, from the point charge +Q is given by,

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a}_r C/m^2 \ldots\ldots\ldots (4)$$

2.5.2 Relationship between $\vec{D}$ and $\vec{E}$

It has been derived that the electric field intensity $\vec{E}$ at a distance of r, from a point charge +Q is given by,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$

Dividing the equations of $\vec{D}$ and $\vec{E}$ due to a point charge +Q we get,

$$\frac{\vec{D}}{\vec{E}} = \frac{\frac{Q}{4\pi r^2}\vec{a}_r}{\frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r} = \varepsilon_0$$

$\therefore$ $$\vec{D} = \varepsilon_0\vec{E} \quad \text{for free space} \ldots\ldots\ldots (5)$$

If the medium in which charge is located is other than free space having relative permittivity ε_r then,

$$\vec{D} = \varepsilon_0\varepsilon_r\vec{E}$$

i.e.

$$\vec{D} = \varepsilon\vec{E} \ldots\ldots.. (6)$$

Example 3.1 : ***Find $\overline{D}$ in cartesian co-ordinate system at point P (6, 8, – 10) due to a) a point charge of 40 mC at the origin, b) a uniform line charge of $\rho_L = 40\,\mu C/m$ on the z-axis and c) a uniform surface charge of density $\rho_S = 57.2\,\mu C/m^2$ on the plane x = 12 m.***

Solution : a) A point charge of 40 mC at the origin.

P(6, 8, – 10) and O(0, 0, 0)

$$\therefore \quad \bar{r} = (6-0)\bar{a}_x + (8-0)\bar{a}_y + (-10-0)\bar{a}_z$$

$$= 6\bar{a}_x + 8\bar{a}_y - 10\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{(6)^2 + (8)^2 + (-10)^2} = \sqrt{200}$$

$$\therefore \quad \bar{a}_r = \frac{6\bar{a}_x + 8\bar{a}_y - 10\bar{a}_z}{\sqrt{200}}$$

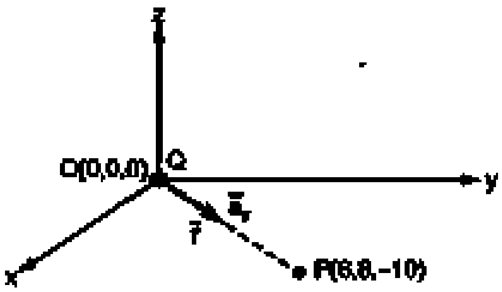

Fig. 3.6

$$\therefore \quad \overline{D} = \frac{Q}{4\pi r^2}\bar{a}_r = \frac{40\times10^{-3}}{4\pi\times(\sqrt{200})^2}\left\{\frac{6\bar{a}_x + 8\bar{a}_y - 10\bar{a}_z}{\sqrt{200}}\right\}$$

$$= 6.752\times10^{-6}\,\bar{a}_x + 9.003\times10^{-6}\,\bar{a}_y - 11.254\times10^{-6}\,\bar{a}_z \ \ C/m^2$$

b) $\rho_l = 40\,\mu C/m$ along z-axis

The charge is infinite hence,

$$\overline{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r$$

As the charge is along z-axis there can not be any component of $\overline{E}$ along z-direction

Consider a point on the line charge (0, 0, z) and P (6, 8, – 10). But while obtaining $\bar{r}$ do not consider z co-ordinate, as $\overline{E}$ and $\overline{D}$ have no $\bar{a}_z$ component.

∴ $\bar{r} = (6-0)\,\bar{a}_x + (8-0)\,\bar{a}_y = 6\,\bar{a}_x + 8\,\bar{a}_y$

∴ $|\bar{r}| = \sqrt{(6)^2 + (8)^2} = 10$

∴ $\bar{a}_r = \dfrac{6\,\bar{a}_x + 8\,\bar{a}_y}{10}$

∴ $\bar{E} = \dfrac{\rho_L}{2\pi\varepsilon_0(10)}\left[\dfrac{6\,\bar{a}_x + 8\,\bar{a}_y}{10}\right]$

∴ $\bar{D} = \varepsilon_0\,\bar{E} = \dfrac{\rho_L}{2\pi\times 10}\left[\dfrac{6\bar{a}_x + 8\bar{a}_y}{10}\right]$

$= 3.819\times10^{-7}\,\bar{a}_x + 5.092\times10^{-7}\,\bar{a}_y\ \ C/m^2$

c) $\rho_S = 57.2\ \mu C/m^2$ on the plane x = 12.

The sheet of charge is infinite over the plane x = 12 which is parallel to yz plane. The unit vector normal to this plane is $\bar{a}_n = \bar{a}_x$.

∴ $\bar{E} = \dfrac{\rho_S}{2\varepsilon_0}\,\bar{a}_n$

The point P is on the back side of the plane hence $\bar{a}_n = -\bar{a}_x$, as shown in the Fig. 3.7.

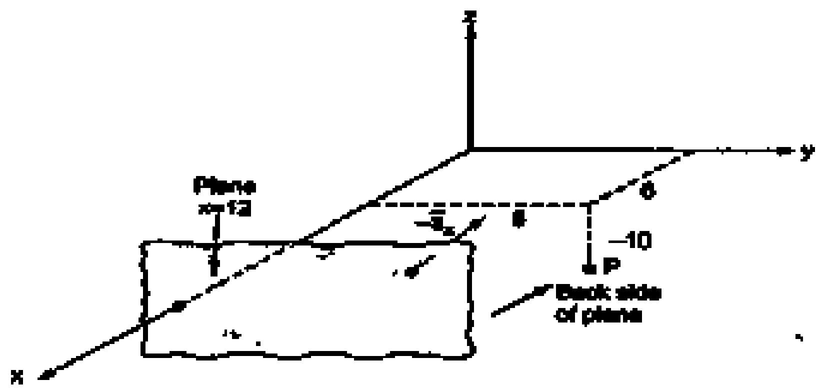

Fig. 3.7

∴ $\bar{E} = \dfrac{\rho_S}{2\varepsilon_0}(-\bar{a}_x)$

But $\bar{D} = \varepsilon_0\bar{E}$

∴ $\bar{D} = \dfrac{\rho_S}{2}(-\bar{a}_x) = -28.6\times10^{-6}\,\bar{a}_x\ \ C/m^2$

2.6 Gauss's law

Statement of Gauss's law:

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

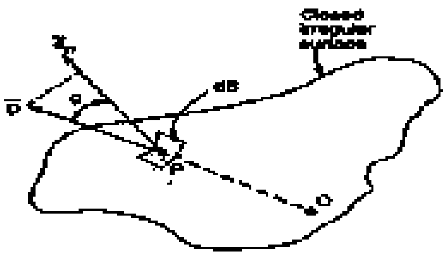

Fig.2.6.1Flux through irregular closed surface

The total charge enclosed by the irregular closed surface is Q coulombs. Consider some small differential surface area dS at point P.

$$\overrightarrow{dS} = dS\ \vec{a}_n$$

Where $\vec{a}_n$ = normal to the surface dS at point P

Key Point: Note that the normal to the surface is in two directions but only directed outwards is considered as required. The normal going into the closed surface at point P is not required.

The flux density at point P is $\vec{D}$ and its direction is such that it makes an angle θ with the normal direction at point P. The flux dΨ passing through the surface dS is the product of the component of $\vec{D}$ in the direction normal to the dS and dS.

Hence $\quad d\Psi = D_n dS$........ (1)

Where $\quad D_n$= Component of $\vec{D}$ in the direction of normal to the surface dS

we can write, $\quad D_n = |\vec{D}|cos\theta$............ (2)

∴ $\quad d\Psi = |\vec{D}|cos\theta dS$ .... (3)

∴ $\quad d\Psi = \vec{D}.\vec{dS}$ .... (4)

Hence the total flux passing through the entire closed surface is to be obtained by finding the surface integration of the equation(4).

∴ $\quad \Psi = \int d\Psi = \oint_S \vec{D}.\vec{dS}$ .... (5)

Such a closed surface over which the integration in the equation (5) is carried out is called **Gaussian Surface**.

Now irrespective of the shape of the surface and the charge distribution, total flux passing through the surface is the total charge enclosed by the surface.

∴ $\quad \Psi = \oint_S \vec{D}.\vec{dS} = Q = \text{charge enclosed}$ .... (6)

The common form used to represent Gauss's law mathematically is,

$$\Psi = Q = \oint_S \vec{D}.\vec{dS} = \int_v \rho_v dv \text{................ (7)}$$

2.6.1 Application of Gauss's Law

The Gauss's law can be used to find $\vec{E}$ or $\vec{D}$ for symmetrical charge distributions, such as point charge, an infinite line charge, an infinite sheet of charge and a spherical distribution of charge. The Gauss's law is also used to find the charge enclosed or the flux passing through the closed surface.

While selecting the closed Gaussian surface to apply the Gauss's law, following conditions must be satisfied,

1. $\vec{D}$ is every where either normal or tangential to the closed surface i.e. $\theta = \frac{\pi}{2}$ or 0. So that $\vec{D}.\vec{dS}$ becomes DdS or zero respectively.
2. $\vec{D}$ is constant over the portion of the closed surface for which $\vec{D}.\vec{dS}$ is not zero.

The steps to obtain $\vec{D}$ and $\vec{E}$ are:

1. Identify $|\vec{D}|$ and its direction.
2. Identify $|\vec{dS}|$ and direction normal to dS.
3. Take dot product, $\vec{D}.\vec{dS}$.
4. Choose the Gussian surface.
5. Integrate over the surface chosen as Gussian surface, keeping $|\vec{D}|$ unknown as it is.
6. Find charge Q enclosed by Gaussian surface.
7. Equate the charge Q, to the integration obtained with $|\vec{D}|$ as unknown.
8. Determine $|\vec{D}|$ and express $\vec{D}$ with its direction. Then $\vec{E} = \vec{D}/\varepsilon_0$.

Let us apply these ideas to the various charge distributions.

2.6.1.1 Point Charge

Let a point charge Q is located at the origin. Consider a Gaussian spherical surface of radius 'r' around Q, with centre as origin. The $\vec{D}$ is always directed radially outwards along $\vec{a}_r$whih is normal to the spherical surface at any point P on the surface. This is shown in Fig.2.6.2.

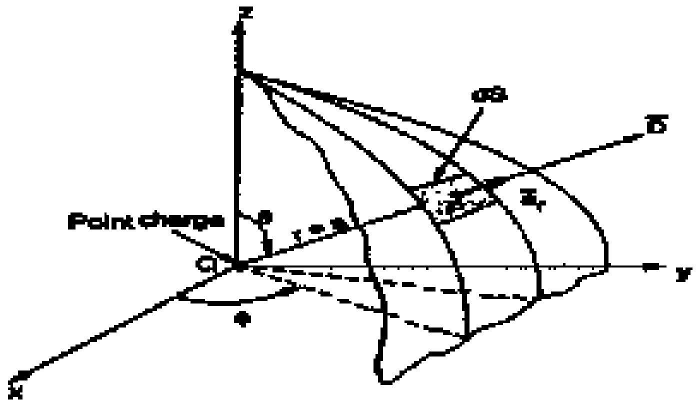

Fig. 2.6.2 Proof of Gauss's law

$$\therefore \qquad \vec{D} = D_r \overrightarrow{a_r}$$

While for the Gaussian surface i.e. sphere of radius 'r', dS normal to $\overrightarrow{a_r}$ is

$$\overrightarrow{dS} = r^2 \sin\theta \, d\theta d\phi \vec{a}_r$$

$$\therefore \qquad \vec{D}.\overrightarrow{dS} = D_r r^2 \sin\theta \, d\theta d\phi \qquad \text{since } \overrightarrow{a_r}.\overrightarrow{a_r} = 1$$

Now integrate over the surface of sphere of constant radius 'r'

$$\oint_S \vec{D}.\overrightarrow{dS} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r r^2 \sin\theta \, d\theta d\phi$$

$$= D_r r^2 [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} = 4\pi r^2 D_r$$

But $\oint_S \vec{D}.\vec{dS} = Q$

∴ $Q = 4\pi r^2 D_r$

∴ $D_r = \frac{Q}{4\pi r^2}$ and hence

And $\vec{D} = D_r \vec{a_r} = \frac{Q}{4\pi r^2} \vec{a}_r$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{Q}{4\pi\varepsilon_0 r^2} \vec{a}_r$$

2.6.1.2 Infinite Line Charge

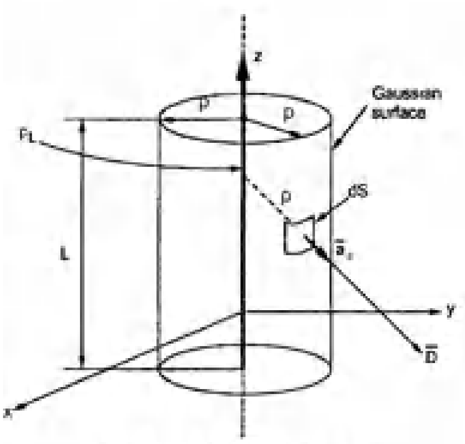

Fig. 2.6.3 Infinite line charge

Consider an infinite line charge of density ρ_L C/m lying along z-axis from $-\infty$ to $+\infty$ as shown in Fig. 2.6.3.

Consider the Gaussian surface as the right circular cylinder with z-axis as its axis and radius ρ. The length of the cylinder is L.

The flux density at any point on the surface is directed radially outwards i.e. in the $\vec{a_\rho}$ direction according to cylindrical co-ordinate system.

Consider differential surface area dS as shown which is at a radial distance ρ from the line charge. The direction normal to dS is $\vec{a_\rho}$.

Now $Q = \oint_S \vec{D}.\vec{dS}$

The integration is to evaluate for side surface, top surface and bottom surface.

∴ $Q = \int_{side} \vec{D}.\vec{dS} + \int_{top} \vec{D}.\vec{dS} + \int_{bottom} \vec{D}.\vec{dS}$

Now $\vec{D} = D_\rho \vec{a}_\rho$ as has only radial component

And $\vec{dS} = \rho d\phi dz\, \vec{a}_\rho$ normal to $\vec{a}_\rho$direction

∴ $\vec{D}.\vec{dS} = D_\rho\, \rho d\phi dz\, (\vec{a}_\rho.\vec{a}_\rho) = D_\rho \rho d\phi dz$

Now D_ρ is constant over the side surface.

As $\vec{D}$ has only radial component and no component along $\vec{a_z}$ and -$\vec{a_z}$ hence integrations over top and bottom surfaces is zero.

∴ $\int_{top} \vec{D}.\vec{dS} = \int_{bottom} \vec{D}.\vec{dS} = 0$

∴ $Q = \int_{side} \vec{D}.\vec{dS} = \int_{side} D_\rho\ \rho d\phi dz$

$$= \int_{z=0}^{L} \int_{\phi=0}^{2\pi} D_\rho\, \rho d\phi dz = \rho D_\rho [z]_0^{L} [\phi]_0^{2\pi}$$

∴ $Q = 2\pi\rho D_\rho L$

∴ $D_\rho = \frac{Q}{2\pi\rho L}$

∴ $\vec{D} = D_\rho \overrightarrow{a_\rho} = \frac{Q}{2\pi\rho L}\vec{a}_\rho$

But $\frac{Q}{L} = \rho_L$ C/m

∴ $\vec{D} = D_\rho \overrightarrow{a_\rho} = \frac{\rho_L}{2\pi\rho}\vec{a}_\rho$ C/m^2… (8) due to infinite line charge

And $\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a}_\rho$ V/m … (9)

The results are same as obtained from the Coulomb's law.

2.6.1.3 Infinite Sheet of Charge

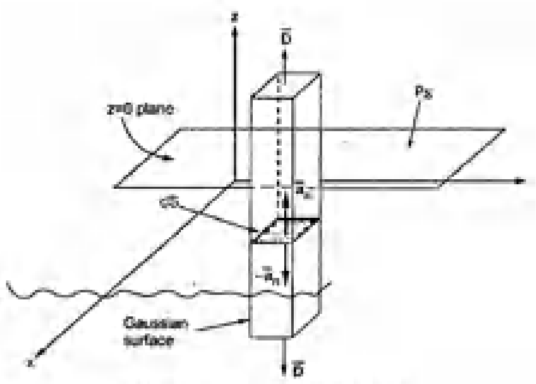

Fig. 2.6.4 Infinite sheet of charge

Consider the infinite sheet of charge of uniform charge density ρ_S C/m^2, lying in the z = 0 plane i.e. xy plane as shown in the Fig. 2.6.4.

Consider a rectangular box as a Gaussian surface which is cut by the sheet of charge to give dS = dx dy.

$\vec{D}$ acts normal to the plane i.e. $\vec{a}_n = \vec{a}_z$ and $-\vec{a}_n = -\vec{a}_z$ direction.

Hence $\vec{D} = 0$ in x and y directions. Hence the charge enclosed can be written as,

∴ $Q = \oint_S \vec{D}.\overrightarrow{dS} = \int_{side} \vec{D}.\overrightarrow{dS} + \int_{top} \vec{D}.\overrightarrow{dS} + \int_{bottom} \vec{D}.\overrightarrow{dS}$

But $\int_{side} \vec{D}.\overrightarrow{dS} = 0$ as $\vec{D}$ has no component in x and y directions.

Now $\vec{D} = D_z\vec{a}_z$ for top surface and $\overrightarrow{dS} = dxdy\vec{a}_z$, hence

$$\vec{D}.\overrightarrow{dS} = D_z dxdy(\vec{a}_z.\vec{a}_z) = D_z dxdy$$

Similarly, $\vec{D} = D_z(-\vec{a}_z)$ for bottom surface and $\overrightarrow{dS} = dxdy(-\vec{a}_z)$, hence

$$\vec{D}.\overrightarrow{dS} = D_z dxdy(-\overrightarrow{a_z}.-\overrightarrow{a_z}) = D_z dxdy$$

∴ $Q = \oint_S \vec{D}.\overrightarrow{dS} = \int_{side} \vec{D}.\overrightarrow{dS} + \int_{top} \vec{D}.\overrightarrow{dS} + \int_{bottom} \vec{D}.\overrightarrow{dS}$

$$= 0 + \int_{top} D_z dxdy + \int_{bottom} D_z dxdy$$

Now $\int_{top} dxdy = \int_{bottom} dxdy = A$ =Area of surface

∴ $Q = 2D_z A$

But $Q = \rho_S \, x \, A$ as ρ_S = surface charge density

Hence $\rho_S = 2D_z$ or $D_z = \frac{\rho_S}{2}$

∴ $\vec{D} = D_z\vec{a}_z = \frac{\rho_S}{2}\vec{a}_z$ C/m²…... (10)

∴ $\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{\rho_S}{2\varepsilon_0}\vec{a}_z$ V/m… (11)

The results are same as obtained from the Coulomb's law for the infinite sheet of charge.

2.6.1.4 Uniformly Charged Sphere

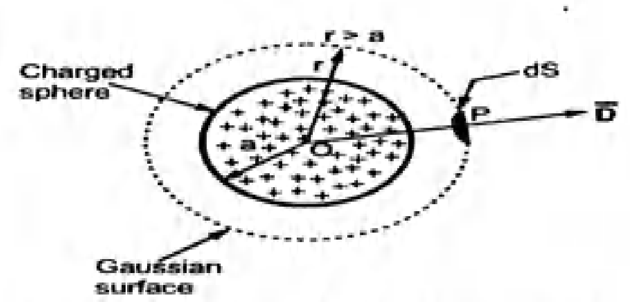

Fig. 2.6.5 Uniformly charged sphere

Consider a sphere of a radius 'a' with a uniform charge density of ρ_v C/m³. Let us find $\vec{E}$ at a point P located at aradial distance r from the centre of sphere such that $r \leq a$ and $r > a$, using Gauss's law. The sphere is shown in Fig. 2.6.5.

Case 1: The point P is outside the sphere ($r > a$)

The Gaussian surface passing through point P is a spherical surface of radius r. The flux lines and $\vec{D}$ are directed radially outwards along $\vec{a_r}$ direction. The differential area dS is considered at point P whose direction is in $\vec{a_r}$ direction i.e normal to the Gaussian surface.

∴ $dS = r^2 sin\theta d\theta d\phi$

∴ $d\Psi = \vec{D}.\vec{dS} = D_r\vec{a}_r.r^2 sin\theta d\theta d\phi\vec{a}_r$

$= D_r r^2 sin\theta d\theta d\phi$ …..$(\vec{a}_r.\vec{a}_r)$

∴ $\Psi = Q = \oint_S \vec{D}.\vec{dS} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} D_r r^2 sin\theta d\theta d\phi$

$$= D_r r^2[-\cos\theta]_0^{\pi}[\phi]_0^{2\pi} = D_r r^2 4\pi$$

∴ $D_r = \frac{Q}{4\pi r^2}$

∴ $\vec{D_r} = \frac{Q}{4\pi r^2}\vec{a}_r$ C/m²………… (12)

∴ $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r$ V/m………….. (13)

Total charge enclosed can be obtained as,

$$Q = \int_v \rho_v dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \rho_v r^2\, sin\theta dr d\theta\, d\phi$$

$$= \rho_v \left[\frac{r^3}{3}\right]_0^a [-cos\theta]_0^\pi [\phi]_0^{2\pi}$$

$$= \frac{4}{3}\pi a^3 \rho_v \text{ C} \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

$$\vec{E} = \frac{\frac{4}{3}\pi a^3 \rho_v}{4\pi \varepsilon_0 r^2} \vec{a}_r = \frac{a^3 \rho_v}{3\varepsilon_0 r^2} \vec{a}_r \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

While $$\overrightarrow{D_r} = \frac{a^3 \rho_v}{3r^2} \vec{a}_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

These are the expressions for $\vec{D}$ and $\vec{E}$ outside the uniformly charged sphere.

Case 2: The point P on the sphere ($r = a$)

The Gaussian surface is same as the surface of the charged sphere. Hence results can be obtained directly substituting r = a in the equation (15) and (16).

∴ $$\vec{E} = \frac{a^3 \rho_v}{3\varepsilon_0 a^2} \vec{a}_r = \frac{a \rho_v}{3\varepsilon_0} \vec{a}_r \ldots\ldots\ldots\ldots\ldots (17)$$

And $$\vec{D} = \varepsilon_0 \vec{E} = \frac{a \rho_v}{3} \vec{a}_r \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

Case 3: The point P is inside the sphere ($r < a$) the Gaussian surface is a spherical surface of radius r where $r < a$

Consider differential surface area dS as shown in the Fig. 2.6.6

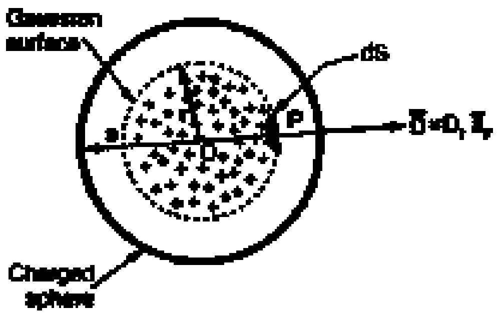

Fig.2.6.6

Again $\overrightarrow{\mathrm{dS}}$ and $\vec{D}$ are directed radially outwards.

∴ $$\vec{D} = D_r \vec{a}_r \text{ while } \overrightarrow{dS} = r^2 sin\theta d\theta d\phi \vec{a}_r$$

∴ $$d\Psi = \vec{\mathrm{D}}.\overrightarrow{\mathrm{dS}} = D_r r^2 sin\theta d\theta d\phi$$

∴ $$\Psi = Q = \oint_S \vec{\mathrm{D}}.\overrightarrow{\mathrm{dS}} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} D_r r^2 sin\theta d\theta d\phi$$

$$= D_r r^2 [-\cos\theta]_0^\pi [\phi]_0^{2\pi} = D_r r^2 4\pi$$

$\therefore \qquad D_r = \frac{Q}{4\pi r^2}$

$\therefore \qquad \overrightarrow{D_r} = \frac{Q}{4\pi r^2}\vec{a}_r$ C/m²………… (19)

Now the charge enclosed is by the sphere of radius r only and not by the entire sphere. The charge outside the Gaussian surface will not affect $\vec{D}$.

$$\therefore \qquad Q = \int_v \rho_v dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} \rho_v r^2 \sin\theta dr d\theta\, d\phi$$

$$= \frac{4}{3}\pi r^3 \rho_v \qquad \text{where } r < a \text{ ……….. (20)}$$

Using in equation (19) we get,

$$\vec{D} = \frac{\frac{4}{3}\pi r^3 \rho_v}{4\pi r^2}\vec{a}_r = \frac{r\rho_v}{3}\vec{a}_r \text{……}0 < r \le a \text{ …………… (21)}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{r\rho_v}{3\varepsilon_0}\vec{a}_r \text{…..….}0 < r \le a \text{…………. (22)}$$

If the sphere is in a medium of permittivity ε_r then ε_0 must be replaced by $\varepsilon = \varepsilon_0\varepsilon_r$.

Variation of $|\vec{E}|$ against r

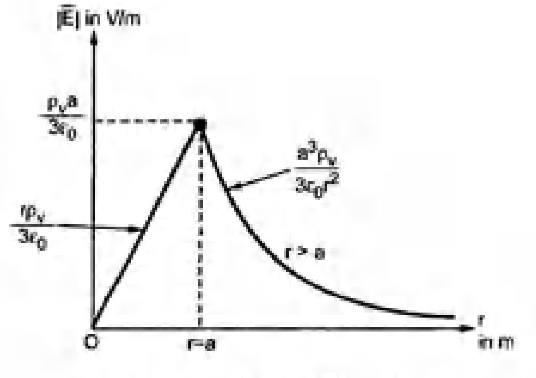

Fig.2.6.7 Variation of $|\vec{E}|$ against r

The graph of $|\vec{D}|$ against r is exactly similar in nature as $|\vec{E}|$ against r.

2.7 Maxwell's First Equation or Point Form of Gauss's law

The divergence of electric flux density $\vec{D}$ is given by,

$$div\,\vec{D} = \lim_{\Delta v \to 0}\frac{\oint_S \vec{D}.\vec{dS}}{\Delta v} \text{………… (1)}$$

According to Gauss's law, it is known that

$$\Psi = Q = \oint_S \vec{D}.\vec{dS} \text{……………. (2)}$$

Expressing Gauss's law per unit volume basis

$$\frac{Q}{\Delta v} = \frac{\oint_S \vec{D}.\overrightarrow{dS}}{\Delta v} \text{............ (3)}$$

Taking lim $\Delta v \to 0$ i.e. volume shrinks to zero,

$$\lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D}.\overrightarrow{dS}}{\Delta v} \text{.......... (4)}$$

But $\lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$ at that point ………….. (5)

The equation (5) gives the volume charge density at that point where divergence is obtained. Equating (1) and (5),

$$div\, \vec{D} = \nabla . \vec{D} = \rho_v \text{......... (6)}$$

This is volume charge density around a point. The equation (6) is called **Maxwell's first equation** applied to electrostatics. This is also called **point form of Gauss's law or Gauss's law in differential form.**

Examples with Solutions

Example 3.10 : *The flux density within the cylindrical volume bounded by r = 5m, z = 0 and z = 2 m is given by,*

$\vec{D} = 30\, r^{-1}\, \bar{a}_r - 2z\, \bar{a}_z \; C/m^2$

What is the total outward flux crossing the surface of the cylinder ?

Solution : The cylinder is shown in the Fig. 3.29.

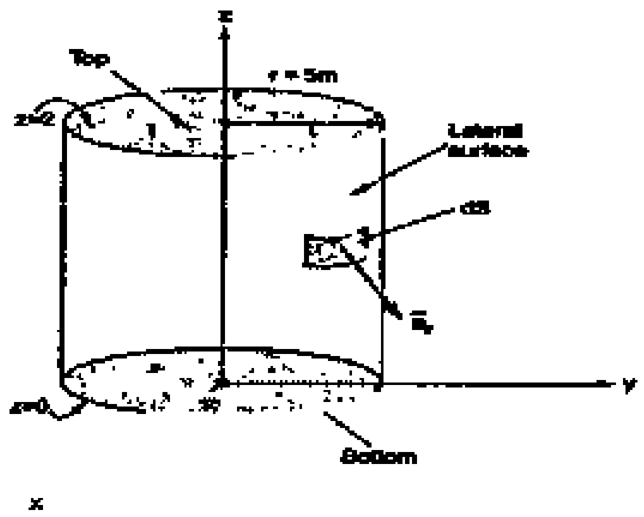

Fig. 3.29

As the total outward flux is asked all surfaces, lateral, top and bottom must be considered.

Case 1 : Consider the lateral surface, the normal direction to which is $\bar{a}_r$.

Consider differential surface area normal to $\bar{a}_r$ which is

$$dS = r\, d\phi\, dz.$$

$$\therefore \quad d\bar{S} = r\, d\phi\, dz\, \bar{a}_r$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = \left[30e^{-r}\,\bar{a}_r - 2z\,\bar{a}_z\right]\cdot\, r\, d\phi\, dz\, \bar{a}_r$$

$$= 30\, r\, e^{-r}\, d\phi\, dz \qquad \ldots (\bar{a}_r\cdot\bar{a}_r = 1,\ \bar{a}_r\cdot\bar{a}_z = 0)$$

According to Gauss's law,

$$\psi_1 = \oint_{\text{lateral}} \bar{D}\cdot d\bar{S} = \int_{z=0}^{2}\int_{\phi=0}^{2\pi} 30\, r\, e^{-r}\, d\phi\, dz \qquad \ldots r = 5 \text{ constant}$$

$$= 30\, r\, e^{-r}\, [\phi]_0^{2\pi}\, [z]_0^{2} \qquad \ldots r = 5 \text{ constant}$$

$$= 30\times 5\times e^{-5}\times 2\pi\times 2 = 12.7 \text{ C}$$

Case 2 : Top surface, for which normal direction is $\bar{a}_z$. The differential area $dS = r\, dr\, d\phi$ normal to $\bar{a}_z$.

$$\therefore \quad d\bar{S} = r\, dr\, d\phi\ \bar{a}_z \quad \text{and} \quad z = 2 \text{ for top surface}$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = \left(30e^{-r}\,\bar{a}_r - 2\,z\,\bar{a}_z\right)\cdot\left(r\, dr\, d\phi\,\bar{a}_z\right)$$

$$= -2\, zr\, dr\, d\phi \qquad \ldots (\bar{a}_z\cdot\bar{a}_z = 1,\ \bar{a}_r\cdot\bar{a}_z = 0)$$

$$\therefore \quad \psi_2 = \oint_{\text{top}} \bar{D}\cdot d\bar{S}$$

$$= \int_{\phi=0}^{2\pi}\int_{r=0}^{5} -2\, z\, r\, dr\, d\phi \quad \text{with } z = 2$$

$$= -2z\left[\frac{r^2}{2}\right]_0^5 [\phi]_0^{2\pi} \qquad \ldots z = 2 \text{ constant}$$

$$= -2\times 2\times 12.5\times 2\pi$$

$$= -314.1592 \text{ C}$$

Case 3 : Bottom surface, for which normal direction is $-\bar{a}_z$ with respect to region. The differential area $dS = r\, dr\, d\phi$ normal to $\bar{a}_z$.

$$\therefore \quad d\bar{S} = r\, dr\, d\phi\,(-\bar{a}_z) \quad \text{and} \quad z = 0 \text{ for bottom}$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = \left(30e^{-r}\,\bar{a}_r - 2z\,\bar{a}_z\right)\cdot\ r\, dr\, d\phi\ (-\bar{a}_z)$$

$$= 2\, zr\, dr\, d\phi \quad \text{with } z = 0$$

$$= 0$$

$$\therefore \quad \psi_3 = \oint_{\text{bottom}} \bar{D}\cdot d\bar{S} = 0 \quad \text{as } z = 0 \text{ for bottom}$$

$$\therefore \quad \psi_{\text{total}} = \psi_1 + \psi_2 + \psi_3$$

$$= -301.4592 \text{ C}$$

Example 3.17 : *A spherical volume charge density is given by,*

$$\rho_v = \rho_0\left(1 - \frac{r^2}{a^2}\right), \quad r \le a$$

$$= 0 \qquad , \quad r > a$$

a) Calculate the total charge Q.

b) Find $\bar{E}$ outside the charge distribution.

c) Find $\bar{E}$ for $r < a$.

d) Show that the maximum value of $\bar{E}$ is at $r = 0.745\, a$. Obtain $|\bar{E}|$ max.

Solution : Note that the ρ_v is dependent on the variable r. Hence though the charge distribution is sphere of radius 'a' we can not obtain Q just by multiplying ρ_v by $\left(\frac{4}{3}\pi a^3\right)$ as ρ_v is not constant. As it depends on r, it is necessary to consider differential volume dv and integrating from $r = 0$ to a, total Q must be obtained. Thus if ρ_v depends on r, do not use standard results.

a) $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$ ... Spherical system

$$\therefore \quad Q = \int_v \rho_v \, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \rho_0\left(1-\frac{r^2}{a^2}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} \int_{r=0}^{a} \left\{ r^2 - \frac{r^4}{a^2} \right\} dr$$

$$= \rho_0 [-(-1)-(-1)][2\pi]\left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]_0^a$$

$$= \rho_0 \times 2 \times 2\pi \times \left[\frac{a^3}{3} - \frac{a^3}{5}\right]$$

$$= \rho_0 \times 4\pi \times \frac{2a^3}{15} = \frac{8\pi}{15}\rho_0 a^3 \text{ C}$$

Outside sphere, $\rho_v = 0$ so $Q = 0$ for $r > a$.

b) The total charge enclosed by the sphere can be assumed to be point charge placed at the centre of the sphere as per Gauss's law.

$$\therefore \quad D = \frac{Q}{4\pi r^2} \bar{a}_r \quad \text{at} \quad r > a$$

$\therefore$ Outside the charge distribution i.e. $r > a$,

$$|E| = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{\frac{8\pi}{15}\rho_0 a^3}{4\pi\varepsilon_0 r^2} = \frac{2}{15}\frac{\rho_0 a^3}{\varepsilon_0}\frac{1}{r^2}$$

$$\therefore \quad \bar{E} = \frac{2}{15}\frac{\rho_0 a^3}{\varepsilon_0}\cdot\frac{1}{r^2}\,\bar{a}_r \text{ V/m}$$

Thus $\bar{E}$ varies with r, outside the charge distribution.

c) For $r < a$, consider a Gaussian surface as a sphere r having $r < a$ as shown in the Fig. 3.36.

Consider dS at point P normal to $\bar{a}_r$ direction, as $\bar{D}$ and $\bar{E}$ are in $\bar{a}_r$ direction.

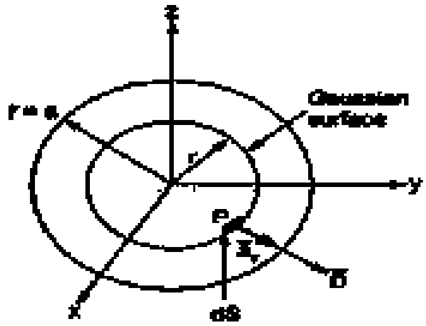

Fig. 3.36

$$d\bar{S} = r^2 \sin\theta \, d\theta \, d\phi \, \bar{a}_r$$

$$\bar{D} = D_r \bar{a}_r$$

$$\therefore \quad \bar{D}\cdot d\bar{S} = D_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$\therefore \quad Q_1 = \oint_S \bar{D}\cdot d\bar{S}$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r \, r^2 [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} = 4\pi r^2 D_r$$

where Q_1 = Charge enclosed by Gaussian surface

$$\therefore \quad D_r = \frac{Q_1}{4\pi r^2}$$

$$\therefore \quad \bar{D} = \frac{Q_1}{4\pi r^2}\bar{a}_r$$

$$\therefore \quad \bar{E} = \frac{\bar{D}}{\varepsilon_0} = \frac{Q_1}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

Let us find Q_1, charge enclosed by Gaussian surface of radius r.

$$Q_1 = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{r} \rho_0\left(1-\frac{r^2}{a^2}\right) r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho_0 [-\cos\theta]_0^{\pi} [\phi]_0^{2\pi} \left\{\frac{r^3}{3} - \frac{r^5}{5a^2}\right\}_0^r$$

$$= 4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right) \text{C}$$

Using in the equation of $\bar{E}$, field intensity for $r < a$ is,

$$\bar{E} = \frac{4\pi\rho_0\left(\frac{r^3}{3}-\frac{r^5}{5a^2}\right)}{4\pi\varepsilon_0 r^2}\bar{a}_r$$

$$= \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3}-\frac{r^3}{5a^2}\right]\bar{a}_r \quad \text{V/m}$$

d) To find $\bar{E}$ to be maximum, inside the sphere i.e. $r < a$ obtain,

$$\frac{d|\bar{E}|}{dr} = 0$$

$$\therefore \frac{d}{dr}\left\{\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3}-\frac{r^3}{5a^2}\right)\right\} = 0$$

$$\therefore \quad \frac{1}{3}-\frac{3r^2}{5a^2} = 0 \qquad \text{as } \rho_v \neq 0,\ \varepsilon_0 \neq 0$$

$$\therefore \quad r^2 = \frac{5a^2}{9}$$

$$\therefore \quad r = 0.745\,a$$ ... Proved

$$\therefore \quad |\bar{E}|_{max} = \frac{\rho_0}{\varepsilon_0}\left[\frac{0.745\,a}{3}-\frac{(0.745\,a)^3}{5a^2}\right]$$

$$= \frac{0.1656\,a\rho_0}{\varepsilon_0} \quad \text{V/m}$$

Example 3.18 : *Three point charges are located in air : + 0.008 μC at (0, 0)m, + 0.005 C at (3, 0)m, and at (0, 4) m there is a a charge of − 0.009 μC. Compute total flux over a sphere of 5 m radius with centre (0, 0).* (UPTU : 2005-06)

Solution : The arrangement is shown in the Fig. 3.36.

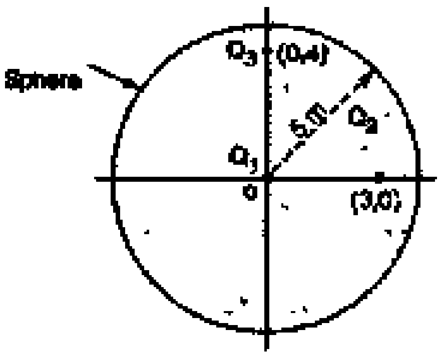

Fig. 3.36

The sphere encloses all the point charges.

$$\therefore \quad Q_{encl} = Q_1 + Q_2 + Q_3$$

$$= 0.008 + 0.008 - 0.009$$

$$= 0.008\ \mu C$$

According to Gauss's law,

$$\psi = Q_{encl} = 0.004\ \mu C$$

This is the flux over a sphere.

2.7 Work Done in Moving a Charge in an Electric Field

Work is said to be done when the test charge is moved against the electric field.

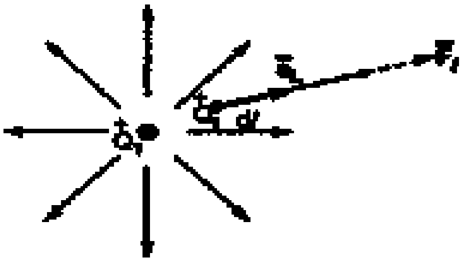

Fig. 2.7.1

Consider a positive test charge Q_1 and its electric field $\vec{E}$. If a positive test charge Q_t is placed in this field, it will move due to the force of repulsion. Let the movement of charge Q_t is $d\vec{l}$. The direction in which the movement has taken place is denoted by unit vector $\overrightarrow{a_l}$, in the direction of $d\vec{l}$, as shown in Fig. 2.7.1.

According to Coulomb's law the force exerted by the field $\vec{E}$ is given by,

$$\vec{F} = Q_t\vec{E} \text{ Newton}\ldots\ldots\ldots\ldots\ldots (1)$$

To keep the charge in equilibrium, it is necessary to apply the force which is equal and opposite to the force exerted by the field in the direction *dl*.

$$\therefore \quad \vec{F}_{applied} = -\vec{F} = -Q_t\vec{E} \quad \text{N} \ldots\ldots\ldots\ldots (2)$$

Key Point: Thus keeping the charge in equilibrium means we are moving a charge Q_t, through the distance $d\vec{l}$ in opposite direction to that of field $\vec{E}$. Hence the work is done.

Hence mathematically the differential work done by an external source in moving the charge Q_t through the distance $d\vec{l}$, against the direction of field $\vec{E}$ is given by,

$$dW = \vec{F}_{applied}.d\vec{l} = -Q_t\vec{E}.d\vec{l}\ldots\ldots\ldots\ldots (3)$$

$$\therefore \quad dW = -Q_t\vec{E}.d\vec{l}\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Thus if a charge Q is moved from initial position to the final position, against the direction of electric field $\vec{E}$ then the total work done is obtained by integrating the differential work done over the distance from initial position to the final position.

$$\therefore \quad W = \int_{initial}^{final} dW = \int_{initial}^{final} -Q\vec{E}.d\vec{l}$$

$$\therefore \quad W = -Q\int_{initial}^{final} \vec{E}.d\vec{l} \quad \text{Joule}\ldots\ldots\ldots\ldots (5)$$

Key Point: Note that at both the positions initial and final, the charge Q is at rest and not moving then the equation (5) is valid.

Consider the charge is moved from initial position B to final position A, against the electric field $\vec{E}$ then work done is given by,

$$W = -Q\int_{B}^{A} \vec{E}.d\vec{l}$$

Thus **the work done in moving a charge from one location B to another A, in a static, uniform or non-uniform electric field $\vec{E}$, is independent of the path selected**. The line integral of $\vec{E}$ is determined completely by the endpoints B and A of the path and not the actual path selected.

Key Point: This is called conservative property of electric field $\vec{E}$ and field $\vec{E}$ is said to be conservative.

2.7.1 Important Comments about Work Done

The work done in moving a point charge in an electric field $\vec{E}$ from position B to A is given by,

$$W = -Q\int_{B}^{A} \vec{E}.d\vec{l}$$

1. When the movement of the charge Q is **against the direction of $\vec{E}$**, then the **work done is positive**, which indicates external source has done the work.
2. When the movement of the charge Q is **in the direction of $\vec{E}$**, then the **work done is negative,** which indicates field itself has done the work, no external source is required.
3. The work done is **independent of the path selected** from B to A but it depends on end points B and A.
4. When the **path selected** is such that it is always **perpendicular to $\vec{E}$** i.e. the force on the charge is always exerted at right angles to the direction in which charge is moving, then the **work done is zero**. This indicates θ, the angle between $\vec{E}$ and $d\vec{l}$ is 90^0. Due to the dot product, the line integral is zero when $\theta = 90^0$.
5. lf the path selected is such that it is forming a **closed contour** i.e. starting point is same as the terminating point then the **work done is zero**.

Example 4.2 : *Consider an infinite line charge along z-axis. Show that the work done is zero if a point charge Q is moving in a circular path of radius r_1, centered at the line charge.*

Solution : The line charge along the z-axis and the circular path along which charge is moving is shown in the Fig. 4.4.

The circular path is in xy plane such that its radius is r_1 and centered at the line charge.

Consider cylindrical co-ordinate system where line charge is along z-axis.

The charge is moving in $\bar{a}_\phi$ direction.

$$\therefore \quad d\bar{L} = r\,d\phi\,\bar{a}_\phi$$

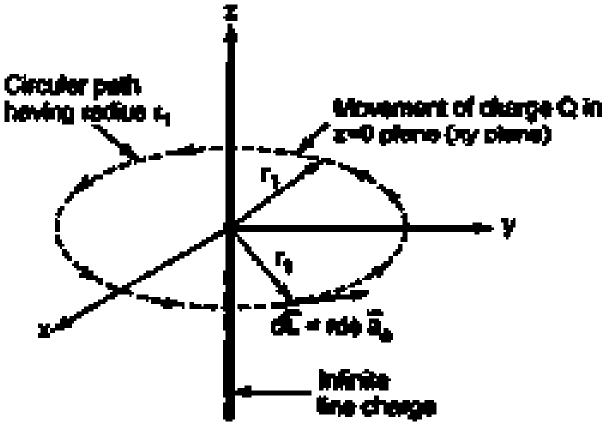

Fig. 4.4

The field $\bar{E}$ due to infinite line charge along z-axis is given in cylindrical co-ordinates as,

$$\bar{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \qquad \text{... (Refer Chapter 2)}$$

The circular path indicates that $d\bar{L}$ has no component in $\bar{a}_r$ and $\bar{a}_z$ direction.

$$\therefore \quad d\bar{L} = r\,d\phi\,\bar{a}_\phi$$

$$\therefore \quad W = -Q\int_{\text{initial}}^{\text{final}} \bar{E}\cdot d\bar{L} = -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\varepsilon_0 r}\bar{a}_r \cdot r\,d\phi\,\bar{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\varepsilon_0}d\phi\,(\bar{a}_r\cdot\bar{a}_\phi) = 0$$

As $\bar{a}_r \cdot \bar{a}_\phi = 0$ as $\theta = 90°$ between $\bar{a}_r$ and $\bar{a}_\phi$.

This shown that the work done is zero while moving a charge such that path is always perpendicular to the $\bar{E}$ direction.

2.8 Potential Difference

Thus the work done in moving a point charge Q from point B to A in the electric field $\vec{E}$ is given by,

$$W = -Q\int_B^A \vec{E}.\overrightarrow{dL} \text{.......... (1)}$$

If the charge Q is selected as unit test charge, then from the above equation we get the work done in moving unit charge from B to A in the field $\vec{E}$. This work done in moving unit charge from point B to A in the field $\vec{E}$ is called potential difference between the points B and A. It is denoted by V.

∴ Potential difference, $V_{AB} = -\int_B^A \vec{E}.\overrightarrow{dL}$……… (2)

Thus work done per unit charge in moving unit charge from B to A in the field $\vec{E}$ is called potential difference between the points B and A.

Key Point: V_{AB} is positive if the work is done by the external source in moving the unit charge from B to A, against the direction of $\vec{E}$.

Hence unit of potential difference is Joules/Coulombs (J/C). But practically the unit is called volt (V). One volt potential difference is one Joule of work done in moving unit charge from one point to other in the field $\vec{E}$.

2.8.1 Potential due to Point Charge

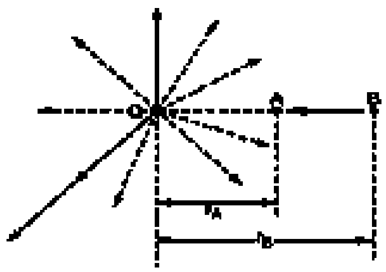

Fig.2.8.1Potential due to a point charge Q

Consider a point charge, located at origin of a spherical co-ordinate system, producing $\vec{E}$ radially in all the directions as shown in Fig, 2.8.1.

Assuming free space, the field $\vec{E}$ due to the point charge Q at a radial distance r from origin is given by,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r \text{……..} (3)$$

Consider a unit charge which is placed at a point B which is at a radial distance of r_B from the origin. It is moved against the direction of $\vec{E}$ from point B to A , which is at a radial distance r_A from the origin. The differential length in spherical system is

$$\overrightarrow{dl} = dr\vec{a}_r + rd\theta\vec{a}_\theta + rsin\theta d\phi\vec{a}_\phi \text{……….. } (4)$$

Hence the potential difference V_{AB} between points A and B is given by,

$$V_{AB} = -\int_B^A \vec{E}.\overrightarrow{dL} \quad \text{But B} = r_B \quad \text{and A} = r_A$$

$$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r.(dr\vec{a}_r + rd\theta\vec{a}_\theta + rsin\theta d\phi\vec{a}_\phi)$$

∴ $$V_{AB} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} \text{dr} \quad \text{……………. } (5)$$

∴ $$V_{AB} = -\frac{Q}{4\pi\varepsilon_0}\int_{r_B}^{r_A} r^{-2}\,\text{dr} = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{r^{-1}}{-1}\right]_{r_B}^{r_A}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_B}^{r_A} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \text{ volt………… } (6)$$

When $r_B > r_A$, $\frac{1}{r_B} < \frac{1}{r_A}$ and V_{AB} is positive. This indicates the work is done by external source in moving unit charge from B to A.

2.8.2 Concept of Absolute Potential

Consider potential difference V_{AB} due to movement of unit charge from B to A in a field of a point charge Q is given by,

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \text{ volt}$$

Now let the charge is moved from infinity to the point A i.e. $r_B = \infty$. Hence $\frac{1}{r_B} = \frac{1}{\infty} = 0$.

$$\therefore \quad V_{AB} = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{\infty}\right] = \frac{Q}{4\pi\varepsilon_0 r_A} \text{ volt} \ldots\ldots\text{ (7)}$$

The quantity represented by equation (7) is called the **potential of point A** denoted as V_A.

$$\therefore \quad V_A = \frac{Q}{4\pi\varepsilon_0 r_A} \text{ volt} \ldots\ldots\ldots\ldots\text{ (8)}$$

Similarly, **absolute potential** of point B can be defined as,

$$\therefore \quad V_B = \frac{Q}{4\pi\varepsilon_0 r_B} \text{ volt} \ldots\ldots\ldots\ldots\text{ (9)}$$

This is work done in moving unit charge from infinity at point B.

Hence the potential difference can be expressed as the difference between the absolute potentials of the two points.

$$\therefore \quad V_{AB} = V_A - V_B \text{ volt} \ldots\ldots\ldots\ldots\text{ (10)}$$

Thus **absolute potential at any point in an electric field is defined as the work done in moving a unit test charge from the infinity (or reference point at which potential is zero) to the point, against the direction of the field**.

Hence absolute potential at any point which is at a distance r from the origin of a spherical system, where point charge Q is located, is given by,

$$V = \frac{Q}{4\pi\varepsilon_0 r} \ldots\ldots\ldots\text{ (11)}$$

The reference point is at infinity.

Key Points: Most widely used reference which is used to develop the concept of absolute potential is infinity. The potential at infinity is treated to be zero and all the potentials at various points in the field are defined with reference to infinity.

2.8.3 Potential due to Point Charge not at Origin

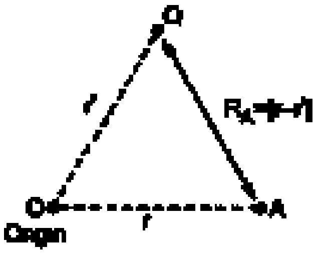

Fig.2.8.3

If the point charge Q is not located at the origin of a spherical system then obtain the position vector r' of the point where Q is located.

Then the absolute potential at a point A located at a distance r from the origin is given by,

$$V(r) = V_A = \frac{Q}{4\pi\varepsilon_0|r-r'|} = \frac{Q}{4\pi\varepsilon_0 R_A} \text{........ (12)}$$

Where $R_A = |r - r'|$ = Distance between point at which potential is to be calculated and the location of the charge

Key Point: R is only the distance and not the vector. The potential is a scalar quantity hence only distance $R_A = |r - r'|$ is involved in the determination of potential of point A. The reference is still infinity.

2.8.4 Potential due to Several Point Charges

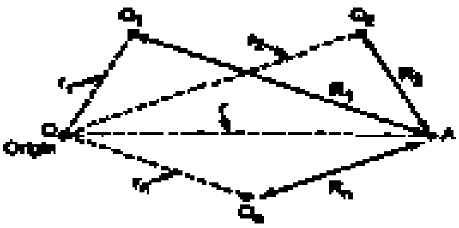

Fig.2.8.4Potential due to several point charges

Consider the various point charges Q_1, Q_2, …Q_n located at the distances r_1, r_2, …., r_n from the origin as shown in the Fig. 2.8.4. The potential due to all these point charges, at point A is to be determined by using Superposition principle.

As the potential is scalar, the net potential at point A is the algebraic sum of the potentials at A due to individual point charges, considered one at a time.

$$\therefore \quad V(r) = V_A = V_{A1} + V_{A2} + \cdots + V_n$$

$$= \frac{Q_1}{4\pi\varepsilon_0 R_1} + \frac{Q_2}{4\pi\varepsilon_0 R_2} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 R_n}$$

$$= \frac{Q_1}{4\pi\varepsilon_0|r-r_1|} + \frac{Q_2}{4\pi\varepsilon_0|r-r_2|} + \cdots + \frac{Q_n}{4\pi\varepsilon_0|r-r_n|}$$

$$\therefore \quad V_A = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0|r-r_m|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 R_m} \text{ volt .... (13)}$$

Example 4.7 : *If three charges, 3 μC, 4 μC and 5 μC are located at (0, 0, 0), (2, − 1, 3) and (0, 4, −2) respectively. Find the potential at (1, 0, 1) assuming zero potential at infinity.*

Solution : Let $Q_1 = 3\mu C$, $Q_2 = -4\mu C$ and $Q_3 = 5\mu C$

The potential of A due to Q_1 is,

$$V_{A1} = \frac{Q_1}{4\pi\varepsilon_0 R_1}$$

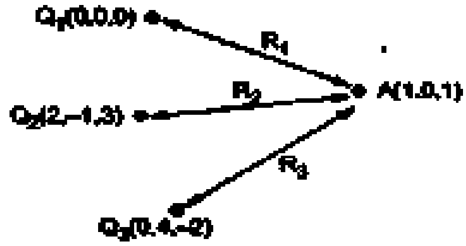

Fig. 4.11

and $R_1 = \sqrt{(1-0)^2+(0-0)^2+(1-0)^2} = \sqrt{2}$

$\therefore$ $V_{A1} = \dfrac{3\times 10^{-6}}{4\pi\varepsilon_0 \times \sqrt{2}}$

$= 19.0658$ kV

The potential of A due to Q_2 is,

$$V_{A2} = \frac{Q_2}{4\pi\varepsilon_0 R_2}$$

and $R_2 = \sqrt{(1-2)^2+[0-(-1)]^2+(1-3)^2} = \sqrt{6}$

$\therefore$ $V_{A2} = \dfrac{-4\times 10^{-6}}{4\pi\varepsilon_0 \times \sqrt{6}} = -14.6769$ kV

The potential of A due to Q_3 is,

$$V_{A3} = \frac{Q_3}{4\pi\varepsilon_0 R_3}$$

and $R_3 = \sqrt{(1-0)^2+(0-4)^2+[1-(-2)]^2} = \sqrt{26}$

$\therefore$ $V_{A3} = \dfrac{5\times 10^{-6}}{4\pi\varepsilon_0 \times \sqrt{26}} = 8.8132$ kV

$\therefore$ $V_A = V_{A1} + V_{A2} + V_{A3} = + 13.2021$ kV

2.8.5 Potential Calculation When Reference is other than Infinity

The expressions derived till now are under the assumption that the reference position of zero potential is at infinity. If any other point than infinity is selected as the reference then the potential at a point A due to point charge Q at the origin becomes,

$$V_A = \frac{Q}{4\pi\varepsilon_0 R_A} + C \text{........ (14)}$$

Where C = Constant to be determined at chosen reference point where V = 0.

Note that the potential difference between the two points is not the function of C.

Key Point: Another important note is that if the potential difference is to be calculated then reference is not needed. The reference is important only when the absolute potential is to be calculated.

Example 4.8 : *A point charge of 6 nC is located at origin in free space, find potential of point P if P is located at (0.2, – 0.4, 0.4) and*

a) V = 0 at infinity

b) V = 0 at (1, 0, 0)

c) V = 20 V at (– 0.5, 1, – 1).

Solution : a) The reference is at infinity, hence

$$V_P = \frac{Q}{4\pi\varepsilon_0 R_P}$$

Fig. 4.12

$$R_P = \sqrt{(0.2-0)^2+(-0.4-0)^2+(0.4-0)^2}$$

$$= 0.6$$

$$\therefore V_P = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times0.6} = 89.8774 \ V$$

b) *V* = 0 at (1, 0, 0). Thus the reference is not at infinity. In such a case potential at *P* is,

$$V_P = \frac{Q}{4\pi\varepsilon_0 R_P}+C$$

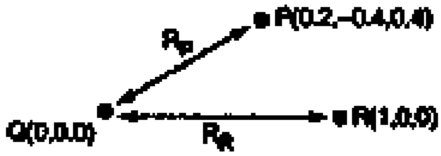

Fig. 4.13

Now V_R at (1, 0, 0) is zero.

$$\therefore \quad V_R = \frac{Q}{4\pi\varepsilon_0 R_R}+C = 0$$

and $$R_R = \sqrt{(1-0)^2+(0)^2+(0)^2} = 1$$

$$\therefore \quad 0 = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times1}+C$$

$$\therefore \quad C = -53.9264$$

$$\therefore \quad V_P = \frac{Q}{4\pi\varepsilon_0 R_P}+C = 89.8774-53.9264 = 35.9509 \ V$$

This is with reference to (1, 0, 0) where V = 0 V.

c) Now V = 20 V at (– 0.5, 1, –1). Let this point is M (– 0.5, 1, – 1). The reference is not given as infinity.

$$V_M = \frac{Q}{4\pi\varepsilon_0 R_M}+C$$

and $$V_M = 20 \ V$$

while $$R_M = \sqrt{(-0.5)^2+(1)^2+(-1)^2} = 1.5$$

Fig. 4.14

$$\therefore \quad 20 = \frac{6\times10^{-9}}{4\pi\times8.854\times10^{-12}\times1.5}+C$$

$$\therefore \quad C = -15.9509$$

$$\therefore \quad V_P = \frac{Q}{4\pi\varepsilon_0 R_P}+C = 89.8774-15.9509$$

$$\therefore \quad V_P = 73.9264 \ V$$

Key Point: *Note that distance of P from origin where Q is located is* R_P *which is same in all the cases. Only 'C' changes as the reference changes hence* V_P *changes.*

Potential difference	$V_{AB} = -\int_B^A \bar{E} \cdot dL$
Absolute potential due to point charge	$V_A = \frac{Q}{4\pi\varepsilon_0 R}$ V
Absolute potential due to line charge	$V_A = \int_L \frac{\rho_L dL'}{4\pi\varepsilon_0 R}$ V
Absolute potential due to surface charge	$V_A = \int_S \frac{\rho_S dS'}{4\pi\varepsilon_0 R}$ V
Absolute potential due to volume charge	$V_A = \int_V \frac{\rho_v dv'}{4\pi\varepsilon_0 R}$ V
If the reference is other than infinity	$V_A = \frac{Q}{4\pi\varepsilon_0 R} + C$ V
In all the expressions, R is the distance of point A from the charge Q or differential charge dQ.	

2.8.6 Potential Difference due to Infinite Line Charge

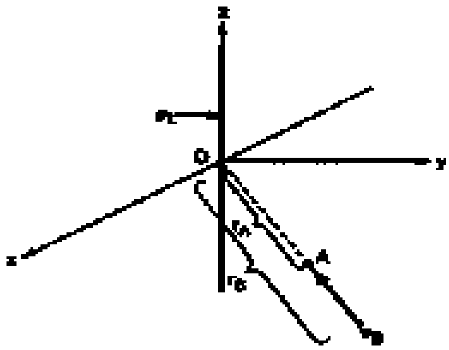

Fig. 2.8.5

Consider an infinite line charge along z-axis having uniform line charge density ρ_L C/m.

The point B is at a radial distance r_B while point A is at a radial distance r_A from the charge, as shown in the Fig. 2.8.5.

The $\vec{E}$ due to infinite line charge along z-axis is known and given by,

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\vec{a}_r$$

While $\overrightarrow{dL} = dr\overrightarrow{a_r}$ in cylindrical system in radial direction.

$$\therefore \quad V_{AB} = -\int_B^A \vec{E}.\overrightarrow{dL} = -\int_{r_B}^{r_A} \frac{\rho_L}{2\pi\varepsilon_0 r}\vec{a}_r . dr\vec{a}_r$$

$$= -\int_{r_B}^{r_A} \frac{\rho_L}{2\pi\varepsilon_0 r} dr = -\frac{\rho_L}{2\pi\varepsilon_0}\int_{r_B}^{r_A} \frac{1}{r} dr = -\frac{\rho_L}{2\pi\varepsilon_0}[\ln r]_{r_B}^{r_A}$$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}[\ln r_A - \ln r_B]$$

$$\therefore \quad V_{AB} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{r_B}{r_A} \text{ volt} \ldots\ldots\ldots\ldots. (15)$$

Example 4.12 : *A line y = 1, z = 1 carries a uniform charge of 2 nC/m, find potential at A(5,0,1) if*
i) V = 0 V at 0 (0, 0, 0) ii) V = 100 V at B (1, 2, 1).

Solution : i) The line is shown in the Fig. 4.23, which is parallel to x-axis.

As reference is not at infinity, to find potential at A means potential of A with respect to origin 0.

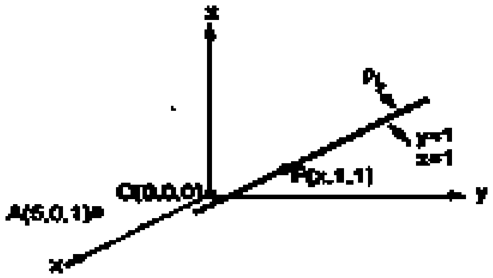

Fig. 4.23

$$\therefore V_{A0} = \frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{r_0}{r_A}\right]$$

where r_A = Perpendicular distance of A from line

$$= \sqrt{(1-0)^2+(1-1)^2} = 1 \text{ m}$$

As line is parallel to x-axis, x co-ordinate is not considered.

and $r_0 = \sqrt{(1-0)^2+(1-0)^2} = \sqrt{2}$ m

$$\therefore V_{A0} = \frac{2\times10^{-9}}{2\pi\times8.854\times10^{-12}} ln\frac{\sqrt{2}}{1} = +12.4596 \text{ V}$$

$$V_{A0} = V_A - V_0$$

$$\therefore 12.4596 = V_A - 0 \quad \text{... as } V_0 = 0 \text{ V}$$

$$\therefore V_A = 12.4596 \text{ V} \quad \text{... This is potential of A}$$

ii) Now V = 100 V at B (1, 2, 1)

$$\therefore V_{AB} = -\frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{r_B}{r_A}\right]$$

where $r_B = \sqrt{(1-2)^2+(1-1)^2} = 1$ ... x is not considered

$$\therefore V_{AB} = -\frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{1}{1}\right] = 0 \text{ V}$$

Now $V_{AB} = V_A - V_B$

$$\therefore 0 = V_A - 100$$

$$\therefore V_A = 100 \text{ V} \quad \text{... This is potential of A}$$

Example 4.13 : *Two uniform line charges, 8 nC/m are located at x = 1, z = 2 and at x = − 1, y = 2 in free space. If the potential at origin is 100 V, find V at P (4, 1, 3).*

Solution : The two line charges are shown in the Fig. 4.24.

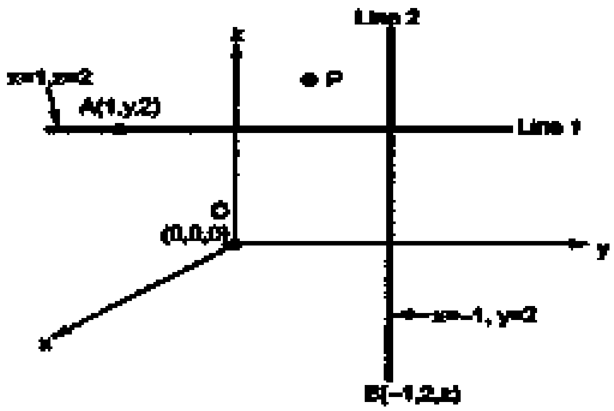

Fig. 4.24

Now V = 100 V at the origin O (0, 0, 0).

Let us obtain potential difference V_{PO} using standard result.

Case 1 : Line charge 1

$$\therefore \quad V_{PO1} = +\frac{\rho_L}{2\pi\varepsilon_0} \ln\left[\frac{r_{O1}}{r_{P1}}\right]$$

where r_{O1} and r_{P1} are perpendicular distances of points O and P from the line 1. The line 1 is parallel to y-axis so do not use y co-ordinates to find r_{O1} and r_{P1}.

$$\therefore \quad r_{O1} = \sqrt{(1-0)^2+(2-0)^2} = \sqrt{5}$$

$$\therefore \quad r_{P1} = \sqrt{(1-4)^2+(2-3)^2} = \sqrt{10}$$

$$\therefore \quad V_{PO1} = +\frac{\rho_L}{2\pi\varepsilon_0} \ln\left[\frac{\sqrt{5}}{\sqrt{10}}\right] = -49.8386$$

$$\text{But} \quad V_{PO1} = V_{P1} - V_O \qquad \text{where} \quad V_O = 100 \text{ V}$$

$$\therefore \quad -49.8386 = V_{P1} - 100$$

$$\therefore \quad V_{P1} = 50.16 \text{ V} \qquad \text{... Absolute potential of P due to line charge 1}$$

Case 2 : Line charge 2, which is parallel to x-axis.

Do not consider x co-ordinate to find perpendicular distance.

$$\therefore \quad r_{O2} = \sqrt{(-1-0)^2+(2-0)^2} = \sqrt{5}$$

$$\text{and} \quad r_{P2} = \sqrt{(-1-4)^2+(2-1)^2} = \sqrt{26}$$

$$\therefore \quad V_{PO2} = \frac{\rho_L}{2\pi\varepsilon_0} \ln\left[\frac{\sqrt{5}}{\sqrt{26}}\right] = -118.5417 \text{ V}$$

$$\text{But} \quad V_{PO2} = V_{P2} - V_O \qquad \text{where } V_O = 100 \text{ V}$$

$$\therefore \quad V_{P2} = -118.5417 + 100 = -18.5417 \text{ V}$$

This is absolute potential of P due to line charge 2

$$\therefore \quad V_P = V_{P1} + V_{P2} = 50.16 - 18.5417 = 31.6183 \text{ V}$$

Note : Students can use the method of using consant C to find absolute potential of P due to line charge 1 and line charge 2. Adding the two, potential of P can be obtained. The answer remains same. For reference, the constant $C_1 = C_2 = 215.721$ for both the line charges.

2.8.7 Equipotential Surfaces

An equipotential surface is an imaginary surface in an electric field of a given charge distribution, in which all the points on the surface are at the same electric potential.

The potential difference between any two points on the equipotential surface is always zero. Thus the work done in moving a test charge from one point to another in an equipotential surface is always zero. There can be many equipotential surfaces existing in an electric field of a particular charge distribution.

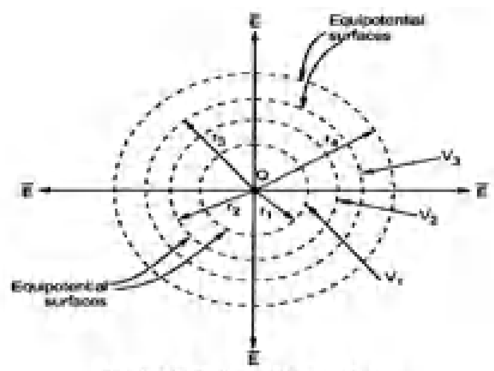

Fig. 2.8.6 Equipotential surfaces

lt can be noted that V is inversely proportional to distance r. Thus V, at equipotential surface at $r = r1$ is highest and it goes on decreasing, as the distance r increases. Thus $V_1 > V_2 > V_3$ As we move away from the charge, the $\vec{E}$ decreases hence potential of equipotential surfaces goes on decreasing. While potential of equipotential surfaces goes on increasing as we move against the direction of electric field.

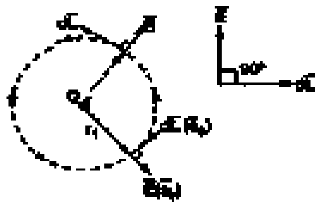
Fig.2.8.7

For a uniform field $\vec{E}$, the equipotential surfaces are perpendicular to $\vec{E}$ and are equispaced for fixed increment of voltages. Thus if we move a charge along a circular path of radius r, as shown in $\overrightarrow{a_\phi}$ direction then work done is zero. This is because $\vec{E}$ and $\overrightarrow{dL}$ are perpendicular. Thus $\vec{E}$ and equipotential surface are at right angles to each other.

2.8.8 Conservative Field

It is seen that, the work done in moving a test charge around any closed path in a static field $\vec{E}$ is zero. This is because starting and terminating point is same for a closed path. Hence upper and lower limit of integration becomes same hence the work done becomes zero. Such an integral over a closed path is denoted as,

$$\oint_L \vec{E}.\overrightarrow{dL} = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Applying Stoke's theorem to equation (16)

$$\oint_L \vec{E}.\overrightarrow{dL} = \int_S (\nabla \times \vec{E}).\overrightarrow{dS} = 0$$

$$\text{Or} \qquad \nabla \times \vec{E} = 0 \ldots\ldots\ldots (17)$$

Key Point: A field having property given by equation (16), associated with it, is called **conservative field or lamellar field**. This indicates that the work done in $\vec{E}$ and hence potential between two points is independent of the path joining the two points. Equation (16) and (17) are referred to as **Maxwell's Equation** in integral and differential form respectively.

2.8.9 Relation between $\vec{E}$ and V

1n space the potential V is unique function of x, y and z co-ordinates, in cartesian system denoted as V (x, y, z). Hence its total differential potential dV can be obtained as,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz \ldots\ldots\ldots (18)$$

In cartesian co-ordinates,

$$\vec{E} = E_x\overrightarrow{a_x} + E_y\overrightarrow{a_y} + E_z\overrightarrow{a_z} \ldots\ldots\ldots (19)$$

While
$$\overrightarrow{dL} = dx\overrightarrow{a_x} + dy\overrightarrow{a_y} + dz\overrightarrow{a_z} \ldots\ldots\ldots (20)$$

∴
$$dV = -\vec{E}.\overrightarrow{dL}$$
$$= -\left[E_x dx + E_y dy + E_z dz\right] \ldots\ldots\ldots (21)$$

Comparing equations (18) and (21) we can write,

$$E_x = -\frac{\partial V}{\partial x}, E_y = -\frac{\partial V}{\partial y}, E_z = -\frac{\partial V}{\partial z} \ldots\ldots\ldots.. (22)$$

Hence $\vec{E}$ can be expressed in terms of (22) as

$$\vec{E} = -\frac{\partial V}{\partial x}\vec{a}_x - \frac{\partial V}{\partial y}\vec{a}_y - \frac{\partial V}{\partial z}\vec{a}_z$$

∴
$$\vec{E} = -\left[\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right]V \ldots\ldots\ldots.. (23)$$

The potential V is scalar but the operator on V given in equation is vector and is called del operator denoted as ∇. The operation of del on a scalar V is called grad V.

∴
$$\nabla = -\left[\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right] \ldots\ldots\ldots.. (24)$$

∴
$$\vec{E} = -\nabla V \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The gradient of a scalar is a vector. The negative sign shows that the direction $\vec{E}$ is opposite to the direction in which V increases i.e. $\vec{E}$ is directed from higher to lower level of V.

➡ **Example 4.18 :** *An electric potential is given by,*

$$V = \frac{60 \sin\theta}{r^2} \text{ V}$$

Find V and $\vec{E}$ at P(3, 60°, 25°).

Solution : At P (3, 60°, 25°), r = 3, θ = 60°, ϕ = 25°

∴
$$V = \frac{60 \sin 60°}{(3)^2} = 5.7735 \text{ V}$$

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi\right)$$

∴
$$\frac{\partial V}{\partial r} = 60 \sin\theta(-2)r^{-3} = -\frac{120 \sin\theta}{r^3} \quad \text{... θ constant}$$

$$\frac{\partial V}{\partial \theta} = \frac{60}{r^2}\cos\theta \quad \text{... r constant}$$

$$\frac{\partial V}{\partial \phi} = 0 \quad \text{... ϕ absent}$$

$$\therefore \qquad \vec{E} = -\left[-\frac{120 \sin \theta}{r^3}\, \bar{a}_r + \frac{1}{r}\frac{60}{r^3}\cos\theta\, \bar{a}_\theta\right]$$

$$\text{At P,} \qquad \vec{E} = -\left[\frac{-120 \sin 60^\circ}{(3)^3}\, \bar{a}_r + \frac{60}{(3)^3}\cos 60^\circ\, \bar{a}_\theta\right]$$

$$= 3.849\, \bar{a}_r - 1.111\, \bar{a}_\theta \quad \text{V/m}$$

2.9 An Electric Dipole

The two point charges of equal magnitude but opposite sign, separated by a very small distance give rise to an **electric dipole**.

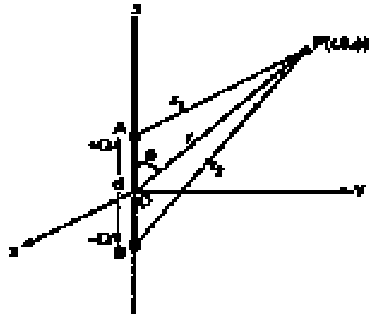

Fig. 2.9.1 Field due to an electric dipole

Consider an electric dipole as shown in the Fig.2.9.1. The two point charges +Q and –Q are separated by a very small distance d.

Consider a point P (r, θ, ϕ) in spherical co-ordinate system. It is required to find $\vec{E}$ due to an electric dipole at point P. Let O be the midpoint of AB. The distance of point P from A is r_1 while the distance of point P from B is r_2. The distance of point P from point O is r. The distance of separation of charges i.e. d is very small compared to the distances r_1, r_2 and r. The co-ordinates of A are $(0,0,+\frac{d}{2})$ and that of B are $(0,0,-\frac{d}{2})$.

2.9.1 Expression of E due to an Electric Dipole

ln spherical co-ordinates, the potential at point P due to the charge + Q is given by,

$$V_1 = \frac{+Q}{4\pi\varepsilon_0 r_1} \quad \ldots\ldots\ldots\ldots (1)$$

The potential at P due to the charge -Q is given by,

$$V_2 = \frac{-Q}{4\pi\varepsilon_0 r_2} \quad \ldots\ldots\ldots\ldots (2)$$

The total potential at point P is the algebraic sum of V_1 and V_2.

$$\therefore \qquad V = V_1 + V_2$$

$$= \frac{+Q}{4\pi\varepsilon_0 r_1} + \frac{-Q}{4\pi\varepsilon_0 r_2}$$

$$\therefore \qquad V = \frac{+Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{+Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \ldots\ldots\ldots\ldots (3)$$

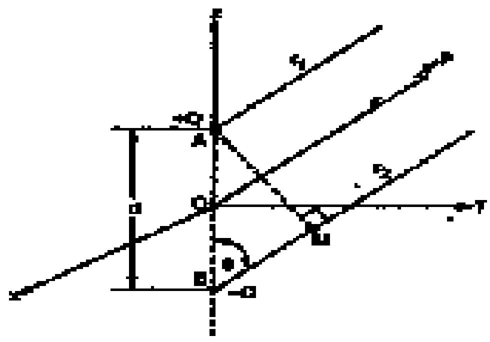

Fig. 2.9.2 Point P is too far away

Now consider that P is located far away from the electric dipole. Thus r_1, r_2 and r can be assumed to be parallel to each other as shown in the Fig. 2.9.2.

As d is very small, $r_1 \approx r_2 \approx r$ hence $r_1 r_2 = r^2$

$\therefore \qquad V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta}{r^2}\right]$ volt

Now $\qquad \vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi\right]$

$\therefore \qquad \frac{\partial V}{\partial r} = \frac{Qd\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right] = \frac{Qd\cos\theta}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial r}(r^{-2})\right]$

$$= \frac{Qd\cos\theta}{4\pi\varepsilon_0}[-2r^{-3}] = \frac{-2Qd\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{Qd}{4\pi\varepsilon_0 r^2}[-\sin\theta] \text{ and } \frac{\partial V}{\partial \phi} = 0$$

$\therefore \qquad \vec{E} = -\left[\frac{-2Qd\cos\theta}{4\pi\varepsilon_0 r^3}\vec{a}_r - \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^2}\vec{a}_\theta\right]$

$\therefore \qquad \vec{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}[2\cos\theta\vec{a}_r + \sin\theta\vec{a}_\theta]\ldots\ldots$ (4) Spherical System

This is electric field $\vec{E}$ at point P due to an electric dipole.

2.9.2 Dipole Moment

Let the vector length directed from — Q to + Q i.e. from B to A is $\vec{d}$.

$\therefore \qquad \vec{d} = d\vec{a}_z \ldots\ldots\ldots\ldots$ (5)

Its component along $\overrightarrow{a_r}$ direction can be obtained as,

$$d_r = \vec{d}.\vec{a}_z = d\vec{a}_z.\vec{a}_r = d\cos\theta$$

$\therefore \qquad \vec{d} = d\cos\theta\vec{a}_r \ldots\ldots\ldots\ldots$ (6)

Then the product Q$\overrightarrow{d}$ is called dipole moment and denoted as $\vec{p}$.

$\therefore \qquad \vec{p} = Q\,\vec{d} \ldots\ldots\ldots\ldots$ (7)

The dipole moment is measured in Cm (coulomb-metre).

Now $\qquad \vec{p}.\vec{a}_r = Q\,\vec{d}.\vec{a}_r = Qd\cos\theta \ldots$ (8)

Hence the expression of potential V can be expressed as,

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}.\vec{a}_r}{4\pi\varepsilon_0 r^2} \text{ volt} \ldots\ldots \text{ (9)}$$

Now if $p = |\vec{p}| = Q|d| = Qd$ then $\vec{E}$ due to a dipole can be expressed in terms of magnitude of dipole moment as,

$\therefore \qquad \vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}[2\cos\theta\vec{a}_r + \sin\theta\vec{a}_\theta]\ldots\ldots$ (10)

Observe that:

1. The potential is inversely proportional to the square of the distance from dipole.

2. The electric field is inversely proportional to the cube of the distance from dipole.

A single point charge is called monopole in which $V \propto \left(\frac{1}{r}\right)$ and $\vec{E} \propto \left(\frac{1}{r^2}\right)$.

Example 4.20 : *A dipole having moment* $\bar{p} = 3\bar{a}_x - 5\bar{a}_y + 10\bar{a}_z$ *nCm is located at* $Q(1, 2, -4)$ *in free space. Find V at* $P(2, 3, 4)$.

Solution : The potential V in terms of dipole moment is,

$$V = \frac{\bar{p} \cdot \bar{a}_r}{4\pi\varepsilon_0 r^2}$$

Now Q (1,2,−4) and P (2,3,4)

$$\therefore \quad \bar{r} = (2-1)\bar{a}_x + (3-2)\bar{a}_y + [4-(-4)]\bar{a}_z$$

$$= \bar{a}_x + \bar{a}_y + 8\bar{a}_z$$

$$\therefore \quad |\bar{r}| = \sqrt{1+1+64} = \sqrt{66}$$

$$\therefore \quad \bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{a}_x + \bar{a}_y + 8\bar{a}_z}{\sqrt{66}}$$

$$\therefore \quad \bar{p} \cdot \bar{a}_r = (3\bar{a}_x - 5\bar{a}_y + 10\bar{a}_z) \cdot \frac{(\bar{a}_x + \bar{a}_y + 8\bar{a}_z)}{\sqrt{66}}$$

$$= \frac{3-5+80}{\sqrt{66}} = \frac{78}{\sqrt{66}} \times 10^{-9} \text{ as } \bar{p} \text{ in nCm}$$

$$\therefore \quad V = \frac{\bar{p} \cdot \bar{a}_r}{4\pi\varepsilon_0 r^2} = \frac{78/\sqrt{66} \times 10^{-9}}{4\pi \times 8.854 \times 10^{-12} \times (\sqrt{66})^2}$$

$$= 1.3074 \text{ V}$$

2.10 Energy Density in the Electrostatic Fields

To determine the energy present in an assembly of charges, first determine the amount of work necessary to assemble them. Let's position three point charges Q_1, Q_2, Q_3 in an initially empty space where there is no electric field at all.

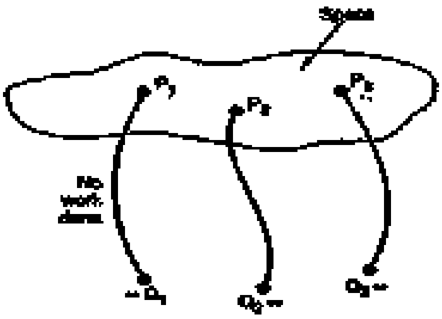

Fig.2.10.1

The charge Q_1 is moved from infinity to a point in the space say P_1. This requires no work as there is no $\vec{E}$ present. Now the charge Q2 is to be placed at point P_2 in the space as shown in the Fig.2.10.1. But now there is an electric field due to Q_1 and Q_2 is required to be moved against the field of Q_1. Hence the work is required to be done.

∴ Work done to position Q_2 at $P_2 = V_{2,1}\, Q_2$………. (1)

Where $V_{2,1}$ = potential at P_2 due to P_1

Now let charge Q_3 is to be moved from infinity to P3. There are electric fields due to Q_1 and Q_2. Hence total work done is due to potential at P_3 due to charge at P_1 and potential at P_3 due to charge at P_2.

Work done to position Q_3 at $P_3 = V_{3,1}Q_3 + V_{3,2}\, Q_3$ …….. (2)

Hence the total work done in positioning all the charges is,

$W_E = Q_2 V_{2,1}+Q_3 V_{3,1}+Q_3 V_{3,2}$ .... (3)

The total work done is nothing but the potential energy in the system of charges hence denoted as W_E. If charges are placed in reverse order we can write,

$W_E = Q_2 V_{2,3}+Q_1 V_{1,2}+Q_1 V_{1,3}$……….. (4)

In this expression Q_3 is placed first, then Q_2 and finally Q_1.

Adding equation (3) and equation (4),

$2W_E = Q_1 V_{1,2}+Q_1 V_{1,3} + Q_2 V_{2,1}+ Q_2 V_{2,3}+Q_3 V_{3,1}+Q_3 V_{3,2}$ ……. (5)

Each sum of the potentials is the total resultant potential due to all the charges except for the charge at the point at which potential is obtained.

∴ $V_{1,2}+ V_{1,3} = V_1$

This is potential at P_1 where Q_1 is placed due to all other charges Q_2, Q_3.

Similarly, $V_{2,1}+ V_{2,3}= V_2$ and so on.

Using in the equation (5),

$2W_E = Q_1 V_1+ Q_2 V_2+Q_3 V_3$……. (6)

If there are ‘n’ point charges, then the potential energy stored in the system is given as,

$$W_E = \frac{1}{2}\sum_{m=1}^{n} Q_m V_m \quad \text{Joule} \ldots\ldots (7)$$

For line charge ρ_L, $W_E = \frac{1}{2}\int \rho_L dLV$ Joule ……….. (8)

For surface charge ρ_s, $W_E = \frac{1}{2}\int \rho_s dSV$ Joule ……….. (9)

For volume charge ρ_v, $W_E = \frac{1}{2}\int \rho_v dvV$ Joule ……….. (10)

2.10.1 Energy Stored ln terms of $\vec{D}$ and $\vec{E}$

Consider the volume charge distribution having uniform charge density ρ_v C/m3. Hence the total energy stored is given by the equation (10) as,

$$W_E = \frac{1}{2}\int_v \rho_v V dv$$

According to Maxwell's first equation,

$$\rho_v = \nabla.\vec{D}$$

$$\therefore \quad W_E = \frac{1}{2}\int_v (\nabla.\vec{D}) V dv \ldots\ldots\ldots\ldots (11)$$

For any vector $\vec{A}$ and scalar V there is vector identity,

$$\nabla.V\vec{A} = \vec{A}.\nabla V + V(\nabla.\vec{A}) \ldots\ldots\ldots (12)$$

$$(\nabla.\vec{A})V = \nabla.V\vec{A} - \vec{A}.\nabla V \ldots\ldots\ldots (13)$$

Using equation (13) in equation (11) we get,

$$W_E = \frac{1}{2}\int_v (\nabla.V\vec{D} - \vec{D}.\nabla V) dv$$

$$\therefore \quad W_E = \frac{1}{2}\int_v (\nabla.V\vec{D}) dv - \frac{1}{2}\int_v (\vec{D}.\nabla V) dv \ldots\ldots\ldots\ldots (14)$$

According to divergence theorem, volume integral can be converted to closed surface integral if closed surface totally surrounds the volume.

$\therefore \qquad \frac{1}{2}\int_v(\nabla . V\vec{D})dv = \frac{1}{2}\oint_S(V\vec{D}).\overrightarrow{dS}$.......... (15)

$\therefore \qquad W_E = \frac{1}{2}\oint_S(V\vec{D}).\overrightarrow{dS} - \frac{1}{2}\int_v(\vec{D}.\nabla \mathrm{V})dv$........... (16)

We know that $V \propto \frac{1}{r}$ and $\vec{E} \propto \frac{1}{r^2}$ for point charge, $V \propto \frac{1}{r^2}$ and $\vec{E} \propto \frac{1}{r^3}$ for dipoles and so on. So $V\vec{D}$ is proportional to at least $\frac{1}{r^3}$ while $\overrightarrow{dS}$ varies as r^2. Hence total integral varies as $\frac{1}{r}$. As surface becomes very large, $r \to \infty$ and $\frac{1}{r} \to 0$ Hence closed surface integral is zero in the equation (16).

$\therefore \qquad W_E = -\frac{1}{2}\int_v(\vec{D}.\nabla \mathrm{V})dv$........... (17)

But $\qquad \vec{E} = -\nabla V$

$\therefore \qquad W_E = -\frac{1}{2}\int_v\left(\vec{D}.\left(-\vec{\mathrm{E}}\right)\right)dv$........... (18)

$\therefore \qquad W_E = \frac{1}{2}\int_v(\vec{D}.\vec{\mathrm{E}})dv$ Joule........... (19)

Now $\qquad \vec{D} = \varepsilon_0\vec{E}$

$\therefore \qquad W_E = \frac{1}{2}\int_v(\varepsilon_0\vec{E}.\vec{E})dv$ Joule

$\therefore \qquad W_E = \frac{1}{2}\int_v(\varepsilon_0 E^2)dv$ Joule .... As $\vec{E}.\vec{E} = E^2$...... (20)

In differential form,

$$dW_E = \frac{1}{2}\,\vec{D}.\vec{E}\,dv$$

$\therefore \qquad \frac{dW_E}{dv} = \frac{1}{2}\,\vec{D}.\vec{E}$ J/m^3............ (21)

This is called **energy density** in the electric field having units J/m^3. If this is integrated over the volume, we get total energy present.

$$W_E = \frac{1}{2}\int_v\left(\frac{dW_E}{dv}\right)dv \text{ .... (22)}$$

Example 4.17 : *If $V = x - y + xy + z$ V, find $\bar{E}$ at (1, 2, 4) and the electrostatic energy stored in a cube of side 2 m centered at the origin.* **[UPTU : 2008-09, 2007-08, 5 Marks]**

Solution : $\qquad V = x - y + xy + z$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$

$$\frac{\partial V}{\partial x} = 1 + y,\ \frac{\partial V}{\partial y} = -1 + x,\ \frac{\partial V}{\partial z} = 1$$

$\therefore \qquad \bar{E} = -[(1 + y)\bar{a}_x + (x - 1)\bar{a}_y + \bar{a}_z]$

At (1, 2, 4); $\qquad \bar{E} = -3\bar{a}_x - \bar{a}_z$ V/m

Now $\qquad W_E = \frac{1}{2}\int_{vol}\varepsilon_0|\bar{E}|^2 dv$, $dv = dx\,dy\,dz$

$$|\bar{E}|^2 = (1 + y)^2 + (x - 1)^2 + 1^2 = 1 + 2y + y^2 + x^2 - 2x + 1 + 1$$
$$= x^2 + y^2 - 2x + 2y + 3$$

$\therefore \qquad W_E = \frac{\varepsilon_0}{2}\int_{vol}(x^2 + y^2 - 2x + 2y + 3)\,dx\,dy\,dz$

The cube is centered at origin hence all the variables x, y and z vary from − 1 to + 1.

$$\therefore \quad W_E = \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\int_{x=-1}^{1}(x^2+y^2-2x+2y+3)\,dx\,dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\left[\frac{x^3}{3}+xy^2-\frac{2x^2}{2}+2xy+3x\right]_{x=-1}^{1}dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\int_{y=-1}^{1}\left[\frac{2}{3}+2y^2+4y+6\right]dy\,dz$$

$$= \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\left[\frac{2}{3}y+\frac{2y^3}{3}+\frac{4y^2}{2}+6y\right]_{y=-1}^{1}dz \;=\; \frac{\varepsilon_0}{2}\int_{z=-1}^{1}\left(\frac{4}{3}+\frac{4}{3}+12\right)dz$$

$$= \frac{\varepsilon_0}{2}\left[\frac{44z}{3}\right]_{-1}^{1} = \frac{88\varepsilon_0}{6} = 0.12985 \text{ nJ}$$

Example 4.18 : *Point charges Q_1 = 1 nC, Q_2 = − 2 nC, Q_3 = 3 nC and Q_4 = − 4 nC are placed one by one in the same order at (0,0,0), (1,0,0), (0,0,−1) and (0,0,1) respectively. Calculate the energy in the system when all charges are placed.*

Solution : When Q_1 is placed, the work done is zero as $\bar{E} = 0$, hence $W_1 = 0$ J.

When Q_2 is placed, there is field of Q_1 present.

$$\therefore \quad W_2 = Q_2 V_{2,1} = Q_2 \times \frac{Q_1}{4\pi\varepsilon_0 R_{21}}$$

and $R_{21} = 1$.

$$\therefore \quad W_2 = \frac{-2\times10^{-9}\times1\times10^{-9}}{4\pi\times8.854\times10^{-12}}$$

$$= -17.9754 \text{ J}$$

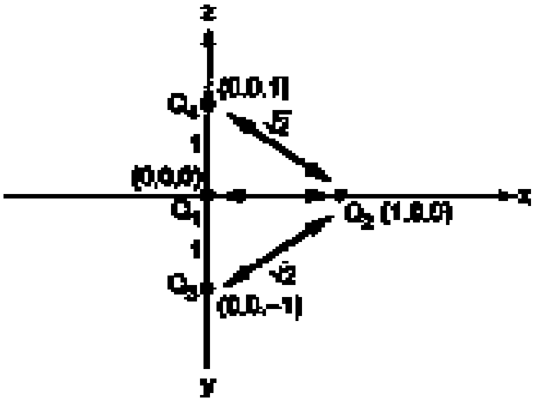

Fig. 4.31

When Q_3 is placed, there is field due to Q_1 and Q_2 both.

$$\therefore \quad W_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$$

$$\therefore \quad W_3 = Q_3\left[\frac{Q_1}{4\pi\varepsilon_0 R_{31}}+\frac{Q_2}{4\pi\varepsilon_0 R_{32}}\right] \quad \text{and } R_{31} = 1,\; R_{32} = \sqrt{2}$$

$$= \frac{3\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{1\times10^{-9}}{1}-\frac{2\times10^{-9}}{\sqrt{2}}\right] = -11.168 \text{ J}$$

When Q_4 is placed, there is field due to Q_1, Q_2 and Q_3.

$$\therefore \quad W_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$$

$$= Q_4\left[\frac{Q_1}{4\pi\varepsilon_0 R_{41}}+\frac{Q_2}{4\pi\varepsilon_0 R_{42}}+\frac{Q_3}{4\pi\varepsilon_0 R_{43}}\right]$$

and $R_{41} = 1$, $R_{42} = \sqrt{2}$, $R_{43} = 2$

$$= \frac{-4\times10^{-9}}{4\pi\varepsilon_0}\left[\frac{1\times10^{-9}}{1}-\frac{2\times10^{-9}}{\sqrt{2}}+\frac{3\times10^{-9}}{2}\right] = -39.035 \text{ J}$$

$$\therefore \quad W_E = W_1 + W_2 + W_3 + W_4 = 0 - 17.9754 - 11.168 - 39.035$$

$$= -68.178 \text{ J}$$

2.11. Current and Current Density

The **current** is defined as the rate of flow of charge and is measured in amperes.

Key Point: A current of 1 ampere is said to be flowing across the surface when a charge of one coulomb is passing across the surface in one second.

The current is considered to be the motion of the positive charges. The conventional current is due to the flow of electrons, which are negatively charged. Hence the direction of conventional current is assumed be opposite to the direction of flow of the electrons The current which exists in the conductors, due to the drifting of electrons, under the influence of the applied voltage is called **drift current**.

While in dielectrics, there can be flow of charges, under the influence of the electric field intensity. Such a current is called the **displacement current or convection current.** The current flowing across the capacitor, through the dielectric separating its plates is an example of the convection current.

The **current density** is defined as the current passing through the unit surface area, when the surface is held normal to the direction of the current. The current density is a vector quantity associated with the current and denoted as $\vec{J}$. The current density is measured in amperes per square metres (A/m^2).

2.11.1 Relation between l and J

Consider a surface S and I is the current passing through the surface. The direction of current I normal to the surface S and hence direction of $\vec{J}$ is also normal to the surface S.

Hence $$I = \int_S J dS \ldots\ldots. (1)$$

where J = Current density in A/m^2.

But if J is not normal to the differential area dS then the total current is obtained by integrating the incremental current which is dot product of J and dS, over the surface S.

Thus in general,

$$I = \int_S \vec{J}.\overrightarrow{dS} \quad \text{(Dot Product)} \ldots\ldots. (2)$$

Thus if J is in A/m^2 and dS is in m^2 then the current obtained is in amperes (A). It may be noted that $\vec{J}$ need not be uniform over S and S need not be a plane surface.

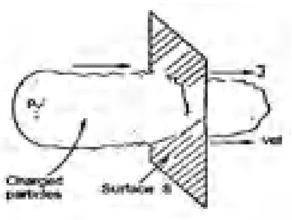

Fig. 2.11.1

In general, the relation between $\vec{J}$ and ρ_v can be expressed as,

$$\vec{J} = \rho_v \vec{v} \ldots\ldots\ldots\ldots (3)$$

where $\vec{v}$= Velocity vector with which charge is getting transferred.

Such a current is called **convection current** and the current density is called **convection current density.**

Key Point: The convection current does not involve conductors and hence does not satisfy Ohm's law. It occurs when current flows through an insulating medium such as liquid, rarefied gas or a vacuum. The convection current density is linearly proportional to the charge density and the velocity with which the charge is transferred.

2.11.2 Continuity Equation

The continuity equation of the current is based on the principle of conservation of charge. The principle states that The charges can neither be created nor be destroyed.

Consider a closed surface S with a current density $\vec{J}$, then the total current I crossing the surface S is given by,

$$I = \oint_S \vec{J}.\overrightarrow{dS} \text{ .... (4)}$$

The current flows outwards from the closed surface. It has been mentioned earlier that the current means the flow of positive charges. Hence the current *I* is constituted due to outward flow of positive charges from the closed surface S.

Let $Q_i =$ Charge within the closed surface

$-\frac{dQ_i}{dt}$ = Rate of decrease of charge inside the closed surface

The negative sign indicates decrease in charge. Due to principle of conservation of charge, this rate of decrease of charge within a given volume must be equal to the net outward current flow through the surface of the volume. Thus current coming out of the closed surface is

$$\therefore \quad I = \oint_S \vec{J}.\overrightarrow{dS} = -\frac{dQ_i}{dt} \text{ .... (5)}$$

This is the integral form of the continuity equation of the current. The negative sign in the equation indicates outward flow of current from the closed surface. So the equation (5) is indicating **outward flowing current I.**

The point form of the continuity equation can be obtained from the 1`integral form. Using the divergence theorem, convert the surface integral in integral form to the volume integral.

$$\oint_S \vec{J}.\overrightarrow{dS} = \int_v (\nabla.\vec{J})dv \text{ .... (6)}$$

$$\therefore \quad -\frac{dQ_i}{dt} = \int_v (\nabla.\vec{J})dv \text{ .... (7)}$$

But

$$Q_i = \int_v \rho_v dv \text{ .... (8)}$$

Where ρ_v = Volume charge density

$$\therefore \quad \int_v (\nabla.\vec{J})dv = -\frac{d}{dt}\left(\int_v \rho_v dv\right) = -\int_v \frac{\partial \rho_v}{\partial t} dv \text{ .... (9)}$$

For a constant surface, the derivative becomes the partial derivative.

$\therefore \qquad \int_v (\nabla.\vec{J})dv = -\int_v \frac{\partial \rho_v}{\partial t} dv$ ……….. (10)

If the relation is true for any volume, it must be true even for incremental volume Δv.

$$(\nabla.\vec{J})dv = -\frac{\partial \rho_v}{\partial t} dv \text{ ……… (11)}$$

$\therefore \qquad \nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$ …………. (12)

This is the **point form or differential form of the continuity equation of the current.**

For **steady currents** which are not the functions of time, $\frac{\partial \rho_v}{\partial t} = 0$ hence,

$$\nabla.\vec{J} = 0 \text{………..…. (13)}$$

For such currents, the rate of flow of charge remains constant with time. The steady currents have no sources or sinks, as it is constant.

Example 5.2 : ***Find the total current in outward direction from a cube of 1 m, with one corner at the origin and edges parallel to the co-ordinate axes if,***
$J = 2x^2 a_x + 2xy^3 a_y + 2xy a_z$ A/m².

Solution : The cube is shown in the Fig. 5.4.

According to continuity equation,

$$I = \oint_S J \cdot dS = \int_{vol} (\nabla \cdot J)\, dv$$

The cube is a volume hence use volume integral.

$$dv = dx\, dy\, dz \quad \text{and}$$

$$\nabla \cdot J = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$= \frac{\partial [2x^2]}{\partial x} + \frac{\partial [2xy^3]}{\partial y} + \frac{\partial [2xy]}{\partial z}$$

$$= 4x + 6xy^2 + 0 = 4x + 6xy^2$$

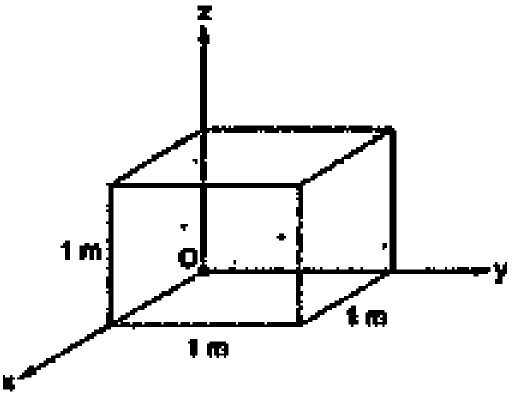

Fig. 5.4

$$\therefore I = \int_{vol} (4x + 6xy^2)\, dx\, dy\, dz = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} (4x + 6xy^2)\, dx\, dy\, dz$$

$$= \int_{z=0}^{1} \int_{y=0}^{1} \left[\frac{4x^2}{2} + \frac{6x^2y^2}{2}\right]_0^1 dy\, dz = \int_{z=0}^{1} \int_{y=0}^{1} (2 + 3y^2)\, dy\, dz$$

$$= \int_{z=0}^{1} \left[2y + \frac{3y^3}{3}\right]_0^1 dz = \int_{z=0}^{1} (2+1)\, dz$$

$$= 3[z]_0^1 = 3 \text{ A}$$

➡ **Example 5.3 :** *A current density* $\bar{J} = \frac{100\cos\theta}{r^2+1}\bar{a}_r\ A/m^2$ *in the spherical co-ordinate system.*

a) How much current flows through the spherical cap r = 3 m, $0<\theta<\frac{\pi}{6}$, $0<\phi<2\pi$

b) The same total current as found in (a) flows through the spherical cap r = 10 m, $0<\theta<\alpha$, $0<\phi<2\pi$. *What should be the value of* α *?*

Solution : a) From the continuity equation of current,

$$I = \oint_S \bar{J}\cdot d\bar{S} = \int_{vol} (\nabla\cdot\bar{J})\, dv$$

As r = 3 m is constant, use surface integral.

$$d\bar{S} = r^2 \sin\theta\, d\theta\, d\phi\, \bar{a}_r \qquad \text{... As } \bar{J} \text{ is in } \bar{a}_r \text{ direction}$$

$$\therefore \quad \bar{J}\cdot d\bar{S} = \frac{100\cos\theta}{(r^2+1)} r^2 \sin\theta\, d\theta\, d\phi\, .$$

$$\therefore \quad \oint_S \bar{J}\cdot d\bar{S} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \frac{100\cos\theta}{(r^2+1)} r^2 \sin\theta\, d\theta\, d\phi$$

$$\therefore \quad I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \frac{100\, r^2}{r^2+1}\times\frac{2\cos\theta\sin\theta}{2}\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/6} \frac{100\, r^2}{r^2+1}\times\frac{\sin 2\theta}{2}\, d\theta\, d\phi \qquad \text{... } 2\sin\theta\cos\theta = \sin 2\theta$$

$$= \frac{100\, r^2}{2(r^2+1)}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/6} [\phi]_0^{2\pi} \qquad \text{and } r = 3\text{ m}$$

$$= \frac{50\times 9}{10}\times\left[\frac{-\cos 2\times\frac{\pi}{6}}{2} - \frac{-\cos 0}{2}\right][2\pi] = 70.6858\text{ A}$$

... Use radian mode to calculate cos

b) Now *r = 10* m and limits for θ are 0 to α.

$$\therefore \quad I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha} \frac{100\, r^2}{(r^2+1)}\,\frac{\sin 2\theta}{2}\, d\theta\, d\phi$$

$$70.6858 = \frac{50\, r^2}{r^2+1}\left[-\frac{\cos 2\theta}{2}\right]_0^{\alpha} [\phi]_0^{2\pi} \qquad \text{... Same I as before}$$

$$\therefore \quad 70.6858 = \frac{50\times(10)^2}{(101)}\times\left[\frac{-\cos 2\alpha}{2} - \frac{-\cos 0}{2}\right][2\pi]$$

$$\therefore \quad 0.4545 = -\cos 2\alpha + 1$$

$$\therefore \quad \cos 2\alpha = 0.5455$$

$$\therefore \quad 2\alpha = 56.9411^\circ \text{ or } 0.9938\text{ rad}$$

$$\therefore \quad \alpha = 28.47^\circ \text{ or } 0.4969\text{ rad}$$

2.11.3 Conductors

The current constituted due to the drifting of electrons under the influence of electric field in metallic conductors is called **drift current**. The drift velocity is directly proportional to the applied electric field.

$$\overrightarrow{v_d} \propto \vec{E} \text{ ………. (14)}$$

The constant of proportionality is called **mobility** of the electrons in a given material and denoted as μ_e. It is positive for the electrons.

∴ $$\overrightarrow{v_d} = -\mu_e \vec{E} \text{ ………. (15)}$$

The negative sign indicates that the velocity of the electrons is against the direction of field $\vec{E}$. Thus mobility is measured in square metres per volt-second (m^2/V. s).

But in the material, the number of protons and electrons is same and it is always electrically neutral. Hence $\rho_v = 0$ for the neutral materials. The drift velocity is the velocity of free electrons hence the above relation can be expressed as,

$$\vec{J} = \rho_e \overrightarrow{v_d} \text{………….. (16)}$$

where ρ_e = Charge density due to free electrons

The charge density ρ_e can be obtained as the product of number of free electrons/m^3 and the charge 'e' on one electron. Thus $\rho_e = ne$ where n is number of free electrons per m^3.

Substituting equation (15) in equation (16) we get,

$$\vec{J} = -\rho_e \mu_e \vec{E} \text{……… (17)}$$

2.11.4 Point Form of Ohm's Law

The relationship between $\vec{J}$ and $\vec{E}$ can also be expressed in terms of conductivity of the material.

Thus for a metallic conductor,

$$\vec{J} = \sigma \vec{E} \text{ …….. (18)}$$

Where $\sigma =$ conductivity of the material

The conductivity is measured in mhos per metre ($\mho/m$). The equation (6) is called **point form of Ohm's law**. The unit of conductivity is also called Siemens per metre (S/ m).

Comparing equation (17) and (18) we get

$$\sigma = -\rho_e \mu_e \text{……… (19)}$$

This is conductivity in-terms of mobility of the charge density of the electrons.

2.11.5 Resistance of a Conductor

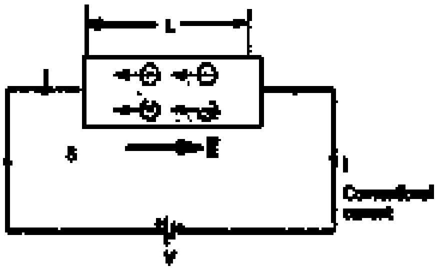

Fig.2.11.2 Conductor subjected to voltage V

Consider that the voltage V is applied to a conductor of length L having uniform cross-section S, as shown in the Fig. 2.11.2.

The direction of $\vec{E}$ is same as the direction of conventional current, which is opposite to the flow of electrons. The electric field applied is uniform and its magnitude is given by,

$$E = \frac{V}{L} \ldots\ldots. (20)$$

The conductor has uniform cross-section S and hence we can write,

$$I = \oint_S \vec{J}.\overrightarrow{dS} = JS \ldots\ldots (21)$$

The current direction is normal to the surface S.

Thus, $$J = \frac{I}{S} = \sigma\vec{E} \ldots\ldots.. (22)$$

And using equation (20) in equation (22) we get,

$$J = \frac{\sigma V}{L} \ldots\ldots\ldots.. (23)$$

$$\therefore \quad V = \frac{JL}{\sigma} = \frac{IL}{S\sigma} = \left(\frac{L}{S\sigma}\right) I \ldots\ldots... (24)$$

$$\therefore \quad R = \frac{V}{I} = \frac{L}{\sigma S} = \frac{\rho_c L}{S} \ldots\ldots... (25)$$

Where $\rho_c =$ Resistivity of the conductor in $\Omega.m$

For non-uniform fields, the resistance R is defined as the ratio V to I where V is the potential difference between two specified equipotential surfaces in the material and I is the current crossing the more positive surface of the two, into the material. Mathematically the resistance for non-uniform fields is given by,

$$R = \frac{V_{ab}}{I} = \frac{-\int_a^b \vec{E}.\overrightarrow{dL}}{\int_s \vec{J}.\overrightarrow{dS}} = \frac{-\int_a^b \vec{E}.\overrightarrow{dL}}{\int_s \sigma\vec{E}.d\vec{S}} \ldots\ldots\ldots (26)$$

2.11.6 Properties of Conductor

Consider that the charge distribution is suddenly unbalanced inside the conductor. There are number of electrons trying to reside inside the conductor. All the electrons are negatively charged and they start repelling each other due to their own electric fields. Such electrons get accelerated away from each other, till all the electrons causing interior imbalance, reach at the surface of the conductor. The conductor is surrounded by the insulating medium and hence electrons just driven from the interior of the conductor, reside over the surface. This,

1. Under static conditions, no charge and no electric field can exist at any point within the conducting material.
2. The charge can exist on the surface of the conductor giving rise to surface charge density.

3. Within a conductor, the charge density is always zero.
4. The charge distribution on the surface depends on the shape of the surface.
5. The conductivity of an ideal conductor is infinite.
6. The conductor surface is an equipotential surface.

Example 6.4 : *A wire of diameter 2 mm and the conductivity 5×10^7 ℧/m has 10^{29} free electrons per m^3. It is subjected to an electric field of 10 mV/m. Determine, a) The free electron charge density b) The current density c) The current in the wire d) The drift velocity of the electrons.*
Given : The charge of one electron = -1.6×10^{-19} C.

Solution : a) $n = 10^{29}$ electrons/m^3 and $e = -1.6\times10^{-19}$ C

$\therefore \quad \rho_e$ = Free electron charge density = ne

$= 10^{29}\times(-1.6\times10^{-19}) = -1.6\times10^{10}\ C/m^3$

b) $J = \sigma E = 5\times10^7\times10\times10^{-3} = 500\ kA/m^2$

c) $I = JS = J\times\frac{\pi}{4}d^2 = 500\times10^3\times\frac{\pi}{4}\times(2\times10^{-3})^2 = 1.5707\ A$

d) $J = \rho_e v_d$

$\therefore \quad 500\times10^3 = -1.6\times10^{10}\, v_d$

$\therefore \quad v_d = -3.125\times10^{-4}$ m/s ... Drift velocity

The negative sign indicates it is opposite to the direction of the applied electric field E.

2.11.7 Relaxation Time

The medium is called **homogeneous** when the physical characteristics of the medium do not vary from point to point but remain same everywhere throughout the medium. If the characteristics vary from point to point. the medium is called **nonhomogeneous** or **heterogeneous**. While the medium is called **linear** with respect to the electric field if the flux density $\vec{D}$ is directly proportional to the electric field $\vec{E}$. The relationship is through the permittivity of the medium. If $\vec{D}$ is not directly proportional to $\vec{E}$, the material is called **nonlinear**.

Consider a conducting material which is linear and homogeneous. The current density for such a material is,

$$\vec{J} = \sigma\vec{E} \quad \text{where } \sigma = \text{conductivity of the material}$$

$$\vec{J} = \sigma\frac{\vec{D}}{\varepsilon} = \frac{\sigma}{\varepsilon}\vec{D} \ldots\ldots (27)$$

The point form of continuity equation states that,

$$\nabla.\vec{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\therefore \quad \nabla.\left(\frac{\sigma}{\varepsilon}\vec{D}\right) = -\frac{\partial \rho_v}{\partial t}$$

$\therefore \qquad \frac{\sigma}{\varepsilon}(\nabla.\vec{D}) = -\frac{\partial \rho_v}{\partial t}$

$But \qquad (\nabla.\vec{D}) = \rho_v$

$\therefore \qquad \frac{\sigma\rho_v}{\varepsilon} = -\frac{\partial \rho_v}{\partial t}$

$\therefore \qquad \frac{\partial \rho_v}{\partial t} + \frac{\sigma\rho_v}{\varepsilon} = 0$……... (28)

This is a differential equation in ρ_v whose solution is given by,

$$\rho_v = \rho_0 e^{-(\sigma/\varepsilon)t} = \rho_0 e^{-t/\tau} \text{…….. (29)}$$

Where ρ_0 = Charge density at t=0

This shows that if there is a temporary imbalance of electrons inside the given material, the charge density decays exponentially with a time constant $\tau = \frac{\varepsilon}{\sigma}$ sec. This time is called **relaxation time**.

The **relaxation time** (τ) is defined as the time required by the charge density to decay to 36.8 % of its initial value.

$$\therefore \qquad \tau = \text{Relaxation time} = \frac{\varepsilon}{\sigma} \text{ sec………… (30)}$$

Key Point: This shows that under static conditions no free charge can remain within the conductor and it gets evenly distributed over the surface of the conductor.

2.12 Boundary Conditions

When an electric field passes from one medium to other medium, it is important to study the conditions at the boundary between the two media. The conditions existing at the boundary of the two media when field passes from one medium to other are called **boundary conditions**. Depending upon the nature of the media, there are two situations of the boundary conditions,

1. Boundary between conductor and free space.
2. Boundary between two dielectrics with different properties.

The free space is nothing but a dielectric hence first case is nothing but the boundary between conductor and a dielectric. For studying the boundary conditions, the Maxwell's equations for electrostatics are required.

$$\oint_L \vec{E}.\overrightarrow{dL} = 0 \; and \; \oint_S \vec{D}.\overrightarrow{dS} = Q \text{……. (1)}$$

Similarly the field intensity $\vec{E}$ is required to be decomposed into two components namely tangential to the boundary ($\vec{E}_{tan}$) and normal to the boundary ($\vec{E}_N$)

$$\therefore \qquad \vec{E} = \vec{E}_{tan} + \vec{E}_N \text{……. (2)}$$

Similar decomposition is required for flux density $\vec{D}$ as well.

2.12.1 Boundary Conditions between Conductor and Free Space

Consider a boundary between conductor and free space. The conductor is ideal having infinite conductivity. Such conductors are copper, silver etc. having conductivity of the order of 10^6 S/ m and can be treated ideal. For ideal conductors it is known that,

1. The field intensity and the flux density inside a conductor is zero.
2. No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density.
3. The charge density within the conductor is zero.

Thus $\vec{E}, \vec{D}$ *and* ρ_v **within the conductor are zero**. While ρ_s is the surface charge density on the surface of the conductor. To determine the boundary conditions let us use the closed path and the Gaussian surface. Consider the conductor free space boundary as shown in the Fig. 2.12.1.

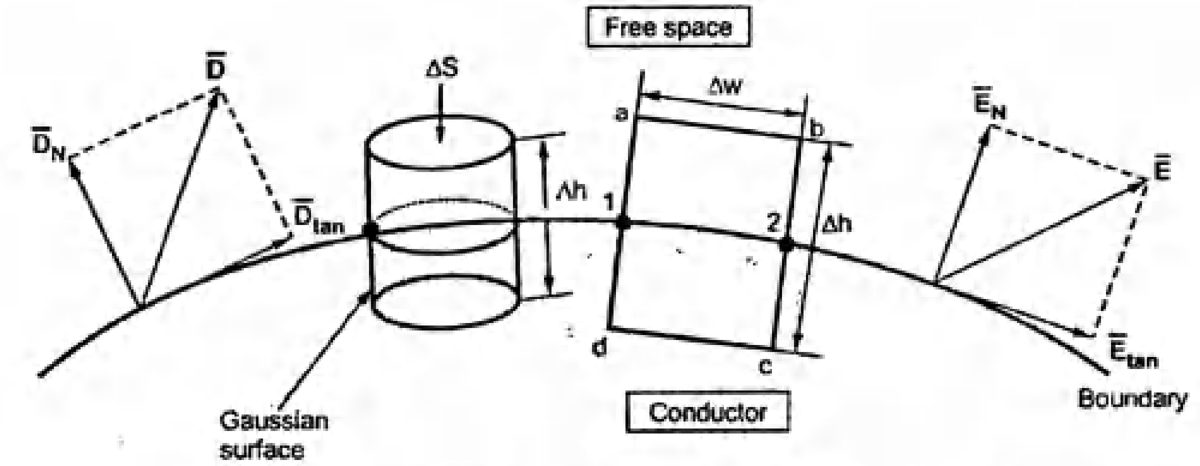

Fig.2.12.2 Boundary between conductor and free space

2.12.2 $\vec{E}$ at the Boundary

Let $\vec{E}$ be the electric field intensity, in the direction shown in the Fig.2.12.1, making some angle with the boundary. This $\vec{E}$ can be resolved into two components :

1. The component tangential to the surface ($\vec{E}_{tan}$).
2. The component normal to the surface ($\vec{E}_N$).

It is known that, $\oint_L \vec{E}.\overrightarrow{dL} = 0$

The integral of $\vec{E}.\overrightarrow{dL}$ carried over a **closed contour** is zero i.e. work done in unit positive charge along a closed path is zero.

Consider a rectangular closed path abcda as shown in the Fig. 2.12.1, It is traced in clockwise direction as a-b-c-d-a and hence $\oint_L \vec{E}.\overrightarrow{dL}$ can be divided into four parts.

$$\oint_L \vec{E}.\overrightarrow{dL} = \int_a^b \vec{E}.\overrightarrow{dL} + \int_b^c \vec{E}.\overrightarrow{dL} + \int_c^d \vec{E}.\overrightarrow{dL} + \int_d^a \vec{E}.\overrightarrow{dL} = 0 \dots\dots\dots (3)$$

The closed contour is placed in such a way that its two sides a-b and c-d are parallel to tangential direction to the surface while the other two are normal to the surface, at the boundary.

The rectangle is an elementary rectangle with elementary height ∇h and elementary width ∇w. The rectangle is placed in such a way that half of it is in the conductor and remaining half is in the free space. Thus $\nabla h/2$ is in the conductor and $\nabla h/2$ is in the free space.

Now the portion c-d is in the conductor where $\vec{E} = 0$ hence the corresponding integral is zero.

$$\therefore \int_a^b \vec{E}.\overrightarrow{dL} + \int_b^c \vec{E}.\overrightarrow{dL} + \int_d^a \vec{E}.\overrightarrow{dL} = 0 \ldots\ldots\ldots (4)$$

As the width ∇w is very small, $\vec{E}$ over it can be assumed constant and hence can be taken out of integration. But ∇w is along tangential direction to the boundary in which direction $\vec{E} = \vec{E}_{tan}$.

$$\therefore \int_a^b \vec{E}.\overrightarrow{dL} = \int_a^b \vec{E}_{tan} . \overrightarrow{dL} = |\vec{E}_{tan}|\nabla w \ldots\ldots (5)$$

Now b-c is parallel to the normal component so we have $\vec{E} = \vec{E}_N$ along this direction. Over the small height ∇h, $\vec{E}_N$ can be assumed constant and can be taken out of integration.

$$\therefore \qquad \int_b^c \vec{E}.\overrightarrow{dL} = \int_b^c \vec{E}_N . \overrightarrow{dL} \ldots\ldots (6)$$

But out of b-c, b-2 is in free space and 2-c is in the conductor where $\vec{E} = 0$.

$$\therefore \qquad \int_b^c \vec{E}_N . \overrightarrow{dL} = |\vec{E}_N| \left(\frac{\nabla h}{2}\right) \ldots\ldots (7)$$

Similarly, for path d-a, the condition is same as for the path b-c, only direction is opposite.

$$\therefore \qquad \int_d^a \vec{E}_N . \overrightarrow{dL} = -|\vec{E}_N| \left(\frac{\nabla h}{2}\right) \ldots\ldots (8)$$

Substituting equations (5), (7) and (8) in (3) we get,

$$\therefore \qquad |\vec{E}_{tan}|\nabla w + |\vec{E}_N| \left(\frac{\nabla h}{2}\right) - |\vec{E}_N| \left(\frac{\nabla h}{2}\right) = 0$$

$$\therefore \qquad |\vec{E}_{tan}|\nabla w = 0 \qquad \text{But } \nabla w \neq 0 \text{ as finite}$$

$$\therefore \qquad |\vec{E}_{tan}| = 0 \ \ldots\ldots (9)$$

Thus the **tangential component of the electric field intensity is zero at the boundary between conductor and free space.**

Key Point: Thus the $\vec{E}$ at the boundary between conductor and free space is always in the direction perpendicular to the boundary.

Now $\qquad \vec{D} = \varepsilon_0 \vec{E}$ for free space

∴ $$\left|\vec{D}_{tan}\right| = \varepsilon_0\left|\vec{E}_{tan}\right| = 0 \ldots\ldots\ldots\ldots \text{(10)}$$

Thus **the tangential component of electric flux density is zero at the boundary between conductor and free space.**

Key Point: Hence electric flux density $\vec{D}$ is also only in the normal direction at the boundary between the conductor and the free space.

2.12.3 $\vec{D}_N$ at the Boundary

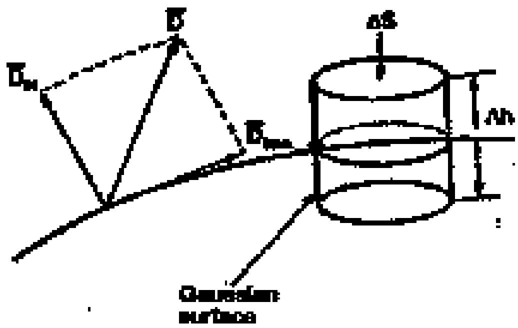

Fig.2.12.3

To find normal component of $\vec{D}$ select a closed Gaussian surface in the form of right circular cylinder as shown in the Fig. 2.12.3. Its height is ∇h and is placed in such a way that $\nabla h/2$ is in the conductor and remaining $\nabla h/2$ is in the free space. Its axis is in the normal direction to the surface.

According to Gauss's law, $\oint_S \vec{D}.\overrightarrow{dS} = Q$

The surface integral must be evaluated over three surfaces,

i) Top, ii) Bottom and iii) Lateral.

Let the area of top and bottom is same equal to ΔS.

∴ $$\int_{top} \vec{D}.\overrightarrow{dS} + \int_{bottom} \vec{D}.\overrightarrow{dS} + \int_{side} \vec{D}.\overrightarrow{dS} = Q \ldots\ldots \text{(11)}$$

The bottom surface is in the conductor where $\vec{D} = 0$ hence corresponding integral is zero. The top surface is in the free space and we are interested in the boundary condition hence top surface can be shifted at the boundary with $\Delta h \to 0$.

∴ $$\int_{top} \vec{D}.\overrightarrow{dS} + \int_{side} \vec{D}.\overrightarrow{dS} = Q \ldots\ldots\ldots \text{(12)}$$

The lateral surface area is $2\pi r\Delta h$ where r is the radius of the cylinder. But as $\Delta h \to 0$, this area reduces to zero and corresponding integral is zero.

While only component of D present is the normal component having magnitude $\left|\vec{D}_N\right|$. The top surface is very small over which D_N can be assumed constant and can be taken out of integration.

∴ $$\int_{top} \vec{D}.\overrightarrow{dS} = \left|\vec{D}_N\right|\Delta S \ldots\ldots\ldots \text{(13)}$$

From Gauss's law,

$$\left|\vec{D}_N\right|\Delta S = Q \ldots\ldots. \text{(14)}$$

But at the boundary, the charge exists in the form of surface charge density ρ_s C/ m^2.

∴ $$Q = \rho_s \Delta S \ldots\ldots\ldots\ldots \text{(15)}$$

Equating equations (14) and (15),

$$\left|\vec{D}_N\right|\Delta S = \rho_s \Delta S$$

$$\left|\vec{D}_N\right| = \rho_s \text{ .... (16)}$$

Thus the **flux leaves the surface normally and the normal component of flux density is equal to the surface charge density.**

$\therefore$ $$\left|\vec{D}_N\right| = \varepsilon_0\left|\vec{E}_N\right| = \rho_s \text{........... (17)}$$

$\therefore$ $$\left|\vec{E}_N\right| = \frac{\rho_s}{\varepsilon_0} \text{ .... (18)}$$

Key Point: Note that as the tangential component of $\vec{E}$ i.e. $\left|\vec{E}_{tan}\right| = 0$, the surface of the conductor is an equipotential surface. The potential difference along any path on the surface of the conductor is $-\int \vec{E}.\overrightarrow{dL}$. and $\vec{E} = \vec{E}_{tan} = 0$ as, the potential difference is zero. Thus all points on the conductor surface are at the same potential.

2.12.4 Boundary Conditions between Conductor and Dielectric

The free space is a dielectric with $\varepsilon = \varepsilon_0$.Thus if the boundary is between conductor and dielectric with $\varepsilon = \varepsilon_0\varepsilon_r$.

$$\left|\vec{D}_{tan}\right| = \left|\vec{E}_{tan}\right| = 0 \text{...... (19)}$$

$$\left|\vec{D}_N\right| = \rho_s \text{.... (20)}$$

$$\left|\vec{E}_N\right| = \frac{\rho_s}{\varepsilon} = \frac{\rho_s}{\varepsilon_0\varepsilon_r} \text{...... (21)}$$

➡ ***Example 5.9 : A potential field is given as $V = 100\, e^{-5x} \sin 3y \cos 4z$ V. Let point P (0.1, $\pi/12$, $\pi/24$) be located at a conductor free space boundary. At point P, find the magnitudes of,***
a) V b) $\bar{E}$ c) E_t d) E_N e) $\bar{D}$ f) D_N g) ρ_s.

Solution : a) At P, $x = 0.1$, $y = \frac{\pi}{12}$, $z = \frac{\pi}{24}$

$\therefore$ $$V = 100\, e^{-0.5} \sin\frac{3\pi}{12} \cos\frac{4\pi}{24} = 37.1422 \text{ V}$$ ... Use radian mode

b) $$\bar{E} = -\nabla V = -\left(\frac{\partial V_x}{\partial x}\bar{a}_x + \frac{\partial V_y}{\partial y}\bar{a}_y + \frac{\partial V_z}{\partial z}\bar{a}_z\right)$$

$$= -100[-5e^{-5x} \sin 3y \cos 4z\,\bar{a}_x + e^{-5x}(3)(\cos 3y)(\cos 4z)\bar{a}_y + e^{-5x}(\sin 3y)(4)(-\sin 4z)\bar{a}_z]$$

At P, $$\bar{E} = [-100\{-(1.657\bar{a}_x + 1.114\bar{a}_y - 0.85776\bar{a}_z]$$

$$= +165.7\,\bar{a}_x - 111.4\,\bar{a}_y + 85.776\,\bar{a}_z \text{ V/m}$$

$\therefore$ $$\left|\bar{E}\right| = 232.9206 \text{ V/m}$$

c) $$E_t = 0 \text{ V/m as P is on the boundary}$$

d) $$E_N = \left|\bar{E}\right| = 232.9206 \text{ V/m}$$

e) $$\bar{D} = \varepsilon_0 \bar{E} = 8.854 \times 10^{-12}[165.7\,\bar{a}_x - 111.4\,\bar{a}_y + 85.776\,\bar{a}_z]$$

$$= 1.466\,\bar{a}_x - 0.9529\,\bar{a}_y + 0.7537\,\bar{a}_z \text{ nC/m}^2$$

$\therefore$ $$\left|\bar{D}\right| = 1.992 \text{ nC/m}^2$$

f) $$D_N = \left|\bar{D}\right| = 1.992 \text{ nC/m}^2$$

g) $$D_N = \rho_s = 1.992 \text{ nC/m}^2$$

2.12.5 Boundary Conditions between Two Perfect Dielectrics

Let us consider the boundary between two perfect dielectrics. One dielectric has permittivity ε_1 while the other has permittivity ε_2. The interface is shown in the Fig. 2.12.4.

The $\vec{E}$ and $\vec{D}$ are to be obtained again by resolving each into two components, tangential to the boundary and normal to the surface.

Consider a closed path abcda rectangular in shape having elementary height Δh and elementary width Δw, as shown in the Fig. 2.12.4. It is placed in such a way that $\Delta h/2$ is in dielectric 1 while the remaining is dielectric 2. Let us evaluate the integral of $\vec{E}.\overrightarrow{dL}$ along this path, tracing it in clockwise direction as a-b-c-d-a.

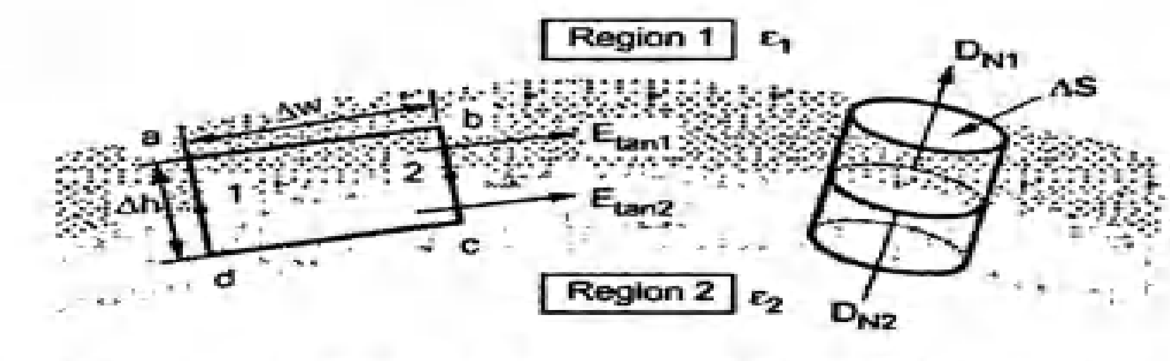

Fig.2.12.4 Boundary between two perfect dielectrics

$$\oint_L \vec{E}.\overrightarrow{dL} = 0 \;\ldots\ldots (22)$$

$$\oint_L \vec{E}.\overrightarrow{dL} = \int_a^b \vec{E}.\overrightarrow{dL} + \int_b^c \vec{E}.\overrightarrow{dL} + \int_c^d \vec{E}.\overrightarrow{dL} + \int_d^a \vec{E}.\overrightarrow{dL} = 0 \ldots.. (23)$$

Now $\quad \vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1N} \ldots\ldots\ldots\ldots (24)$

And $\quad \vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2N} \ldots\ldots\ldots\ldots (25)$

Both $\vec{E}_1$ and $\vec{E}_2$ in the respective dielectrics have both the components, normal and tangential.

Let $\quad |\vec{E}_{1t}| = \vec{E}_{tan1},\ |\vec{E}_{2t}| = \vec{E}_{tan2}$

$\quad |\vec{E}_{1N}| = \vec{E}_{N1},\ |\vec{E}_{2N}| = \vec{E}_{N2}$

Now for the rectangle to be reduced at the surface to analyse boundary conditions, $\Delta h \to 0$. As $\Delta h \to 0$, $\int_b^c$ and $\int_d^a$ become zero as these are line integrals along Δh and $\Delta h \to 0$.

Hence equation (23) reduces to,

$$\int_a^b \vec{E}.\overrightarrow{dL} + \int_c^d \vec{E}.\overrightarrow{dL} = 0 \ldots\ldots (26)$$

Now a-b is in dielectric 1 hence the corresponding component of $\vec{E}$ is $\vec{E}_{tan1}$ as a-b direction is tangential to the surface.

$\therefore \quad \int_a^b \vec{E}.\overrightarrow{dL} = \int_a^b \vec{E}_{tan1}.\overrightarrow{dL} = |\vec{E}_{tan1}|\Delta w \ldots\ldots\ldots (27)$

While c-d is in dielectric 2 hence the corresponding component of $\vec{E}$ is $\vec{E}_{tan2}$ as c-d direction is also tangential to the surface. But direction c-d is opposite to a-b hence corresponding integral is negative of the integral obtained for path a-b.

$\therefore \quad \int_c^d \vec{E}.\overrightarrow{dL} = \int_c^d -\vec{E}_{tan2}.\overrightarrow{dL} = -|\vec{E}_{tan2}|\Delta w$………. (28)

Substituting equation (27) and equation (28) in equation (26) we get,

$$|\vec{E}_{tan1}|\Delta w - |\vec{E}_{tan2}|\Delta w = 0$$

$$|\vec{E}_{tan1}| = |\vec{E}_{tan2}| \text{……… (29)}$$

Thus **the tangential components of field intensity at the boundary in both the dielectrics remain same i.e. electric field intensity is continuous across the boundary.**

The relation between $\vec{D}$ and $\vec{E}$ is known as,

$$\vec{D} = \varepsilon\vec{E} \text{……….. (30)}$$

Hence if $|\vec{D}_{tan1}|$ and $|\vec{D}_{tan2}|$ are magnitudes of the tangential components of $\vec{D}$ in dielectric 1 and 2 respectively then,

$$|\vec{D}_{tan1}| = \varepsilon_1|\vec{E}_{tan1}| \text{ and } |\vec{D}_{tan2}| = \varepsilon_2|\vec{E}_{tan2}| \text{…… (31)}$$

$\therefore \quad \frac{|\vec{D}_{tan1}|}{|\vec{D}_{tan2}|} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$………… (32)

Thus **tangential components of $\vec{D}$ undergoes some change across the interface hence tangential $\vec{D}$ is said to be discontinuous across the boundary**.

To find the normal components, let us use Gauss's law. Consider a Gaussian surface in the form of right circular cylinder, placed in such a way that half of it lies in dielectric 1 while the remaining half in dielectric 2. The height $\Delta h \rightarrow 0$ hence flux leaving from its lateral surface is zero. The surface area of its top and bottom is ΔS.

$\therefore \quad \oint_S \vec{D}.\overrightarrow{dS} = Q$…………… (33)

$\therefore \quad \left[\int_{top} + \int_{bottom} + \int_{side}\right]\vec{D}.\overrightarrow{dS} = Q$……….. (34)

But $\quad \int_{side} \vec{D}.\overrightarrow{dS} = 0 \quad$ as $\Delta h \rightarrow 0$

$\therefore \quad \left[\int_{top} + \int_{bottom}\right]\vec{D}.\overrightarrow{dS} = Q$……….. (35)

The flux leaving normal to the boundary is normal to the top and bottom surfaces.

$\therefore \quad |\vec{D}| = |\vec{D}_{N1}|$ for dielectric 1 and $|\vec{D}_{N2}|$ for dielectric 2.

And as top and bottom surfaces are elementary, flux density can be assumed constant and can be taken out of integration.

$\therefore \quad \int_{top} \vec{D}.\overrightarrow{dS} = \int_{top} \vec{D}_{N1}.\overrightarrow{dS} = |\vec{D}_{N1}|\Delta S$……….. (36)

For top surface, the direction of $\vec{D}_N$ is entering the boundary while for bottom surface, the direction of $\vec{D}_N$ is leaving the boundary. Both are opposite in direction, at the boundary.

$\therefore$ $$\int_{bottom} \vec{D}.\overrightarrow{dS} = \int_{bottom} -\vec{D}_{N2}.\overrightarrow{dS} = -|\vec{D}_{N2}|\Delta S \quad \text{...... (37)}$$

$\therefore$ $$|\vec{D}_{N1}|\Delta S - |\vec{D}_{N2}|\Delta S = Q \quad \text{...... (38)}$$

But $$Q = \rho_s \Delta S \quad \text{................ (39)}$$

$\therefore$ $$|\vec{D}_{N1}| - |\vec{D}_{N2}| = \rho_s \quad \text{........... (40)}$$

There is no free charge available in perfect dielectric hence no free charge can exist on the surface. All charges in dielectric are bound charges and are not free. Hence at the ideal dielectric media boundary the surface charge density ρ_s can be assumed zero.

$\therefore$ $$|\vec{D}_{N1}| = |\vec{D}_{N2}| \quad \text{......... (41)}$$

Hence the normal component of flux density D is continuous at the boundary between the two perfect dielectrics.

$\therefore$ $$\frac{|\vec{D}_{N1}|}{|\vec{D}_{N2}|} = \frac{\varepsilon_1 |\vec{E}_{N1}|}{\varepsilon_2 |\vec{E}_{N2}|} = 1 \qquad \text{As } |\vec{D}_{N1}| = |\vec{D}_{N2}|$$

$\therefore$ $$\frac{|\vec{E}_{N1}|}{|\vec{E}_{N2}|} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} \quad \text{.............. (42)}$$

The normal components of the electric field intensity $\vec{E}$ are inversely proportional to the relative permittivities of the two media.

2.13 Concept of Capacitance

The ratio of the magnitudes of the total charge on any one of the two conductors and potential difference between the conductors is called the **capacitance** of the two conductor system denoted as C.

Hence capacitance can be expressed as,

$$C = \frac{Q}{V} = \frac{\oint_S \vec{D}.\overrightarrow{dS}}{-\int_-^+ \vec{E}.\overrightarrow{dL}} = \frac{\varepsilon \oint_S \vec{E}.\overrightarrow{dS}}{-\int_-^+ \vec{E}.\overrightarrow{dL}} \quad \text{......... (1)}$$

Key Point: The capacitance depends on the physical dimensions of the system and the properties of the dielectric such as permittivity of the dielectric.

Some Important Two Conductor Configurations

2.13.1 Parallel Plate Capacitor

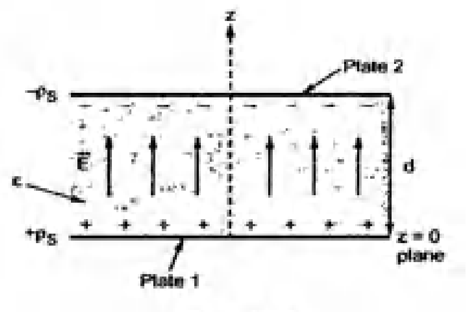

Fig. 2.13.1

A parallel plate capacitor is shown in the Fig. 2.13.1. It consists of two parallel metallic plates separated by distance 'd'. The space between the plates is filled with a dielectric of permittivity ε. The lower plate, plate 1 carries the positive charge and is distributed over it. The upper plate, plate 2 carries the negative charge and is distributed over its surface. The plate 1is placed in z=0 i.e. xy plane hence normal to it is z-direction. While upper plate 2 is in $z = d$ plane, parallel to xy plane.

Let A = Area of cross section of the plates in m^2.

$\therefore \qquad Q = \rho_s A$ Coulomb………. (2)

This is magnitude of charge on any one plate as charge carried by both is equal in magnitude. To find potential difference, let us obtain $\vec{E}$ between the plates.

The space between the plates is filled with homogenous dielectric with permittivity ε, so $\vec{D} = \rho_s \vec{a}_z$ …….(3)

Again $\vec{E} = \frac{\vec{D}}{\varepsilon}$…………….(4)

So $\vec{E} = \frac{\rho_s}{\varepsilon}\vec{a}_z$……. (5)

The potential difference is given by,

$$V = -\int_{-}^{+} \vec{E}.\overrightarrow{dL} = -\int_{upper}^{lower} \frac{\rho_s}{\varepsilon}\vec{a}_z.(dx\overrightarrow{a_x} + dy\overrightarrow{a_y} + dz\overrightarrow{a_z})$$

$$V = -\int_{z=d}^{z=0} \frac{\rho_s}{\varepsilon} dz = -\frac{\rho_s}{\varepsilon}[z]\begin{matrix}0\\d\end{matrix} = -\frac{\rho_s}{\varepsilon}[-d]$$

$$V = \frac{\rho_s d}{\varepsilon} \text{ volt………. (6)}$$

The capacitance is the ratio of charge Q to voltage V,

$$C = \frac{Q}{V} = \frac{\rho_s A}{\frac{\rho_s d}{\varepsilon}} = \frac{\varepsilon A}{d} \text{ Farad………. (7)}$$

Thus if, $\qquad \varepsilon = \varepsilon_0 \varepsilon_r$, $\quad$ then $C = \frac{\varepsilon_0 \varepsilon_r A}{d}$ Farad……….. (8)

Capacitance is not dependant on the charge or the potential difference between the plates rather dependant on permittivity, area of cross section of the plates and separation of the plates.

2.13.2 Capacitance of a Co-Axial Cable

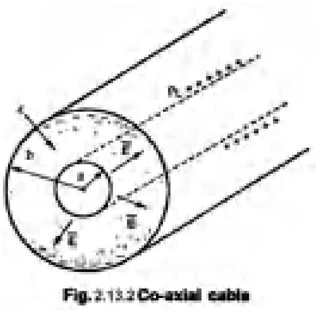

Fig.2.13.2 Co-axial cable

Let a = Inner radius, b = Outer radius

The two concentric conductors are separated by dielectric of permittivity ε. The length of the cable is L meter. The inner conductor carries a charge density $+\rho_L$ C/m on its surface then equal and opposite charge density $-\rho_L$ C/m exists on the outer conductor.

$\therefore \qquad Q = \rho_L \times L$………. (9)

Assuming cylindrical co-ordinate system, $\vec{E}$ will be radial from inner to outer conductor, and for infinite line charge it is given by,

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon r}\vec{a}_r \text{…….. (10)}$$

$\vec{E}$ is directed from inner conductor to the outer conductor. The potential difference is work done in moving unit charge against $\vec{E}$ i.e. from r = b to r = a.

To find potential difference, consider $\overrightarrow{dL}$ in radial direction which is $dr\vec{a}_r$

$\therefore \quad V = -\int_{-}^{+} \vec{E}.\overrightarrow{dL} = -\int_{r=b}^{r=a} \frac{\rho_L}{2\pi\varepsilon r}\vec{a}_r.dr\vec{a}_r$

$$= -\frac{\rho_L}{2\pi\varepsilon}[lnr]_b^a = -\frac{\rho_L}{2\pi\varepsilon} ln\left[\frac{a}{b}\right]$$

$\therefore \quad V = \frac{\rho_L}{2\pi\varepsilon} ln\left[\frac{b}{a}\right]$ volt .... (11)

$\therefore \quad C = \frac{Q}{V} = \frac{\rho_L \times L}{\frac{\rho_L}{2\pi\varepsilon} ln\left[\frac{b}{a}\right]}$

$\therefore \quad C = \frac{2\pi\varepsilon L}{ln\left[\frac{b}{a}\right]}$ F .... (12)

2.13.3Spherical Capacitor

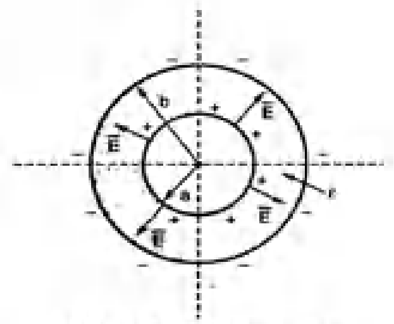

Fig.2.13.3Spherical capacitor

Consider a spherical capacitor formed of two concentric spherical conducting shells of radius ‘a’ and ‘b’. The capacitor is shown in the Fig. 2.13.3.

The radius of outer sphere is 'b' while that of inner sphere is 'a'. Thus b > a. The region between the two spheres is filled with a dielectric of permittivity ε. The inner sphere is given a positive charge $+Q$ while for the outer sphere it is $-Q$.

Considering Gaussian surface as a sphere of radius r, it can be obtained that $\vec{E}$ is in radial direction and given by,

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\vec{a}_r \quad \text{V/m.............. (13)}$$

The potential difference is work done in moving unit positive charge against the direction of $\vec{E}$ i.e. from $r = b$ to $r = a$.

$$V = -\int_{-}^{+} \vec{E}.\overrightarrow{dL} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\vec{a}_r.dr\vec{a}_r$$

$$= -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2} dr = -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_b^a$$

$\therefore \quad V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$ volt......... (14)

Now $\quad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]}$

$\therefore \quad C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]}$ Farad .... (15)

2.13.4 Capacitance of Single Isolated Sphere

Consider a single isolated sphere of radius ‘a’, given a charge $+Q$. It forms a capacitance with an outer plate which is infinitely large hence $b = \infty$.

The capacitance of such a single isolated spherical conductor can be obtained by substituting $b = \infty$ in the equation (15).

$$\therefore \quad C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{\infty}\right]}$$

$$C = 4\pi\varepsilon a \ldots\ldots\ldots (16) \quad \left(\text{as } \frac{1}{\infty} = 0\right)$$

➡ **Example 5.19 :** *A pair of 200 mm long concentric cylindrical conductors of radii 50 mm and 100 mm, is filled with a dielectric with $\varepsilon = 10\varepsilon_0$. A voltage is applied between the conductors which establishes $\bar{E} = \frac{10^6}{r}\bar{a}_r$. Calculate :*

a) Capacitance b) Voltage applied c) Energy stored.

Solution : The arrangement is shown in the Fig. 5.26.

a) The capacitor is not a function of voltage V or $\bar{E}$, it depends on dielectric and physical dimensions. For coaxial conductors,

$$C = \frac{2\pi\varepsilon L}{ln\left(\frac{b}{a}\right)}$$

$$= \frac{2\pi(10\varepsilon_0)\times 200\times 10^{-3}}{ln\left[\frac{100\times 10^{-3}}{50\times 10^{-3}}\right]}$$

$$= 160.518 \text{ pF}$$

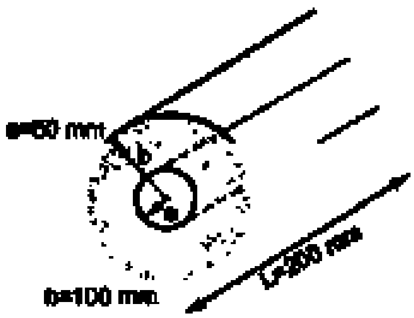

Fig. 5.26

b) $\bar{E}$ is function of r hence using,

$$V = -\int_{-}^{+} \bar{E}\cdot d\bar{L} = -\int_{r=b}^{r=a} \frac{10^6}{r}\bar{a}_r \cdot [dr\,\bar{a}_r]$$

$$= -\int_{r=b}^{r=a} \frac{10^6}{r}\,dr = -10^6\,[ln\,r]_{r=100\text{ mm}}^{r=50\text{ mm}}$$

$$= -10^6\,[ln\,50\times 10^{-3} - ln\,100\times 10^{-3}] = 693.1471 \text{ kV}$$

c)
$$W_E = \frac{1}{2}C\,V^2 = \frac{1}{2}\times 160.518\times 10^{-12}\times(693.147\times 10^3)^2$$

$$= 38.5606 \text{ J}$$

Alternatively,
$$W_E = \int \frac{1}{2}\varepsilon\left|\bar{E}\right|^2 dv = \frac{\varepsilon}{2}\int\left|\frac{10^6}{r}\right|^2 r\,dr\,d\phi\,dz$$

$$= \frac{10^{12}\,\varepsilon}{2}\int_{z=0}^{L}\int_{\phi=0}^{2\pi}\int_{r=a}^{b}\frac{1}{r}\,dr\,d\phi\,dz$$

$$= 38.56 \text{ J}$$

➡ **Example 5.20 :** *A conducting sphere of radius 5 cm has a total charge of 1 μC. The sphere is surrounded by an inhomogeneous dielectric sphere $5 \le r \le 10$ cm in which relative permittivity varies as $\varepsilon_r = 0.1/r$. A second conducting spherical surface is at $r = 10$ cm. Calculate the potential difference and capacitance between the conductors.*

Solution : The arrangement is shown in the Fig. 5.32.

As $\varepsilon_r = \frac{1}{r}$, the standard formula for spherical capacitor cannot be used.

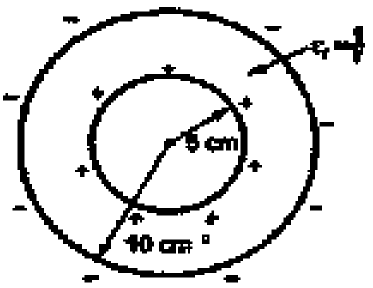

Fig. 5.32

In spherical conductor $\bar{E}$ at a radial distance r is given by,

$$\bar{E} = \frac{Q}{4\pi\varepsilon r^2}\,\bar{a}_r \text{ V/m}$$

$$\therefore \qquad V = -\int_{-}^{+} \bar{E}\cdot d\bar{L}$$

$$= -\int_{r=10\text{ cm}}^{r=5\text{ cm}} \frac{Q}{4\pi\varepsilon r^2}\,\bar{a}_r \cdot dr\,\bar{a}_r \qquad \text{... Note } \varepsilon=\varepsilon_0\varepsilon_r$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_{r=0.1}^{r=0.05} \frac{1}{\left[\frac{0.1}{r}\right] r^2}\,dr = -\frac{Q}{4\pi\varepsilon_0}\int_{r=0.1}^{0.05} \frac{10}{r}\,dr$$

$$= -\frac{10Q}{4\pi\varepsilon_0}\left\{\ln(r)\right\}_{0.1}^{0.05} = -\frac{10Q}{4\pi\varepsilon_0}\ln\left[\frac{0.05}{0.1}\right] \qquad \text{... } Q = 1\ \mu C$$

$$= 62.298 \text{ kV}$$

And $$C = \frac{Q}{V} = \frac{1\times10^{-6}}{62.298\times10^{3}} = 16.051 \text{ pF}$$

➡ **Example 5.22 :** *A spherical condenser has a capacity of 54 pF. It consists of two concentric spheres differing in radii by 4 cm and having air as dielectric. Find their radii.*

Solution : $C = 54$ pF and $b - a = 4$ cm, $\varepsilon_r = 1$

The spherical capacitor has a capacitance,

$$C = \frac{4\pi\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \quad \text{where } b > a$$

$$\therefore \qquad 54\times10^{-12} = \frac{4\pi\times 8.854\times10^{-12}\,(ab)}{(b-a)\times10^{-2}}$$

but $b - a = 4$ cm, 10^{-2} as $b - a$ in cm

$$\therefore \qquad ab = 1.9413\times10^{-2}$$

Now $$b = 4\times10^{-2} + a \qquad \text{... From } b - a = 4 \text{ cm}$$

$$\therefore \qquad a\,(4\times10^{-2}+a) = 1.9413\times10^{-2}$$

$$\therefore \qquad a^2 + 0.04\,a - 0.019413 = 0$$

$$\therefore \qquad a = \frac{-0.04 \pm \sqrt{(0.04)^2 - (4)(-0.019413)}}{2} = 0.1207 \text{ m}$$

... Neglecting negative value

And $$b = 4\times10^{-2} + a = 0.1607 \text{ m}$$

Electrostatic boundary-value problems

2.14 Introduction

In earlier chapters, the $\vec{E}$ and $\vec{D}$ in the given region are obtained using Coulomb's law and Gauss's law. Using these laws is easy, if the charge distribution or potential throughout the region is known. Practically it is not possible in many situations, to know the charge distribution or potential variation throughout the region. Practically charge and potential may be known at some boundaries of the region, only. From those values it is necessary to obtain potential and $\vec{E}$ throughout the region. Such electrostatic problems are called **boundary value problems**. To solve such problems, Poisson's and Laplace's equations must be known.

2.14.1 Poisson's and Laplace's Equations

From the Gauss's law in the point form, Poisson's equation can be derived. Consider the Gauss's law in the point form as,

$$\nabla.\vec{D} = \rho_v \ldots\ldots\ldots (1)$$

where $\vec{D}$ = Flux density and ρ_v = Volume charge density

It is known that for a homogeneous, isotropic and linear medium, flux density and electric field intensity are directly proportional. Thus,

$$\vec{D} = \varepsilon\vec{E} \ldots\ldots\ldots (2)$$

$$\therefore \quad \nabla.\varepsilon\vec{E} = \rho_v \ldots\ldots\ldots (3)$$

From the gradient relationship,

$$\vec{E} = -\nabla V \ldots\ldots.. (4)$$

Substituting (4) in (3),

$$\nabla.\varepsilon(-\nabla V) = \rho_v \ldots\ldots\ldots (5)$$

Taking $-\varepsilon$ outside as constant,

$$-\varepsilon[\nabla.\nabla V] = \rho_v$$

$$\therefore \quad [\nabla.\nabla V] = -\frac{\rho_v}{\varepsilon}$$

Now $\nabla.\nabla$ operation is called 'del squared' operation and denoted as ∇^2.

$$\therefore \quad \nabla^2 V = -\frac{\rho_v}{\varepsilon} \ldots\ldots.. (6)$$

This equation (6) is called **Poisson's equation**.

If in a certain region, volume charge density is zero ($\rho_v = 0$), which is true for dielectric medium then the Poisson's equation takes the form,

$$\nabla^2 V = 0 \text{ (For charge free region)} \ldots\ldots. (7)$$

This is special case of Poisson's equation and is called **Laplace's equation**. The ∇^2 operation is called the **Laplacian of V**.

Key Point: Note that $(\rho_v = 0)$, still in that region point charges, line charges and surface charges may exist at singular locations.

The equation (7) is for homogeneous medium for which ε is constant. But it ε is not constant and the medium is inhomogeneous, then equation (5) must be used as Poisson's equation for inhomogeneous medium.

In Cartesian co-ordinate system, Laplace's equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \ldots\ldots (8)$$

Similarly, in Cylindrical coordinate system, it is given as,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0 \ldots\ldots (9)$$

And in Spherical coordinate system, it is given as,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \ldots\ldots (10)$$

2.14 Uniqueness Theorem

The boundary value problems can be solved by number of methods such as analytical, graphical, experimental etc. Thus there is a question that, is the solution of Laplace's equation solved by any method, unique? The answer to this question is the uniqueness theorem, which is proved by contradiction method.

Assume that the Laplace's equation has two solutions say V_1 and V_2, both are function of the co-ordinates of the system used. These solutions must satisfy Laplace's equation. So we can write

$$\nabla^2 V_1 = 0 \text{ and } \nabla^2 V_2 = 0 \ldots\ldots\ldots.. (1)$$

Both the solutions must satisfy the boundary conditions as well. At the boundary, the potentials at different points are same due to equipotential surface then,

$$V_1 = V_2 \ldots\ldots\ldots (2)$$

Let the difference between the two solutions is V_d.

$$V_d = V_2 - V_1 \ldots\ldots\ldots (3)$$

Using Laplace's equation for the difference V_d,

$$\nabla^2 V_d = \nabla^2 (V_2 - V_1) \ldots\ldots\ldots (4)$$

∴ $$\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 \ldots\ldots.. (5)$$

On the boundary $V_d = 0$ from the equations (2) and (3).

Now the divergence theorem states that,

$$\int_v (\nabla.\vec{A}).dv = \oint_S \vec{A}.\overrightarrow{dS} \ldots\ldots. (6)$$

Let $\vec{A} = V_d \nabla V_d$ and from vector algebra

$$\nabla.(\alpha\vec{B}) = \alpha(\nabla.\vec{B}) + \vec{B}.\nabla\alpha$$

Now use this for $\nabla.(V_d \nabla V_d)$ with $\alpha = V_d$ and $\vec{B} = \nabla V_d$

$$\nabla.(V_d \nabla V_d) = V_d(\nabla.\nabla V_d) + \nabla V_d.\nabla(V_d)$$

But $\nabla.\nabla = \nabla^2$ hence,

$$\nabla.(V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d . \nabla(V_d) \ldots\ldots\ldots (7)$$

Using equation (4),

$$\nabla.(V_d \nabla V_d) = \nabla V_d . \nabla V_d \ldots\ldots\ldots (8)$$

To use this in equation (6), let $\vec{A} = V_d \nabla V_d$ hence

$$\nabla. V_d \nabla V_d = \nabla. \vec{A} = \nabla V_d . \nabla V_d$$

$$\int_v (\nabla V_d . \nabla V_d). dv = \oint_S V_d \nabla V_d . \overrightarrow{dS} \ldots\ldots. (9)$$

But $V_d = 0$ on boundary, hence right hand side of equation (9) is zero

∴ $$\int_v (\nabla V_d . \nabla V_d). dv = 0$$

∴ $$\int_v |\nabla V_d|^2 . dv = 0 \quad \text{as } \nabla V_d \text{ is a vector} \ldots\ldots. (10)$$

Now integration can be zero under two conditions,

i) The quantity under integral sign is zero.

ii) The quantity is positive in some regions and negative in other regions by equal amount and hence zero.

But square term cannot be negative in any region hence, quantity under integral must be zero.

Hence $$|\nabla V_d|^2 = 0$$

i.e. $$\nabla V_d = 0 \ldots\ldots\ldots\ldots\ldots (11)$$

As the gradient of $V_d = V_2 - V_1$ is zero means $V_2 - V_1$ is constant and not changing with any co-ordinates. But considering boundary it can be proved that $V_2 - V_1 = \text{constant} = \text{zero}.$

$$V_1 = V_2 \ldots\ldots\ldots. (12)$$

This proves that both the solutions are equal and cannot be different. Thus **Uniqueness Theorem** can be stated as:

If the solution of Laplace's equation satisfies the boundary condition then that solution is unique, by whatever method it is obtained.

2.14.1 Procedure for Solving Laplace's Equation

The procedure to solve a problem involving Laplace's equation can be generalized as,

Step 1: Solve the Laplace's equation using the method of integration. Assume constants of integration as per the requirement.

Step 2: Determine the constants applying the boundary conditions. Given constants obtained using boundary conditions is a unique solution.

Step 3: Then $\vec{E}$ can be obtained for the potential field V obtained, using gradient operation $-\nabla V$.

Step 4: For homogeneous medium, $\vec{D}$ can be obtained as $\varepsilon\vec{E}$.

Step 5: At the surface, $\rho_S = \vec{D}_N$ hence once $\vec{D}$ is known, the normal component $\vec{D}_N$ to the surface is known. Hence the charge induced on the conductor surface can be obtained as $Q = \int_S \rho_S . \vec{dS}$

Step 6: Once the charge induced Q is known and potential V is known then the capacitance C of the system can be obtained.

If $\rho_v \neq 0$ then similar procedure can be adopted to solve the Poisson's equation.

➡ **Example 4.1 :** ***Determine whether or not the following potential fields satisfy the Laplace's equation :***

a) $V = x^2 - y^2 + z^2$ *b)* $V = r\cos\phi + z$ *c)* $V = r\cos\theta + \phi$ **(UPTU : [illegible])**

Solution : a) $V = x^2 - y^2 + z^2$

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2}{\partial x^2}[x^2 - y^2 + z^2] + \frac{\partial^2}{\partial y^2}[x^2 - y^2 + z^2] + \frac{\partial^2}{\partial z^2}[x^2 - y^2 + z^2]$$

$$= \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial y}[-2y] + \frac{\partial}{\partial z}[2z] = 2 - 2 + 2 = 2$$

So $\nabla^2 V \neq 0$

Hence field V does not satisfy Laplace's equation.

b) $V = r\cos\phi + z$

In cylindrical co-ordinate system,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}[r\cos\phi + z] = \cos\phi$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}[r\cos\phi + z] = -r\sin\phi$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}[r\cos\phi + z] = 1$$

$$\therefore \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}[r\cos\phi] = \frac{1}{r}\cos\phi$$

$$\frac{1}{r^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial \phi}(-r\sin\phi)\right] = -\frac{r\cos\phi}{r^2} = -\frac{\cos\phi}{r}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}[1] = 0$$

$$\therefore \nabla^2 V = \frac{1}{r}\cos\phi - \frac{\cos\phi}{r} + 0 = 0$$

So this field satisfies Laplace's equation.

c) $V = r\cos\theta + \phi$

In spherical system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$r^2\frac{\partial V}{\partial r} = r^2\frac{\partial}{\partial r}[r\cos\theta + \phi] = r^2(\cos\theta)$$

$$\sin\theta\frac{\partial V}{\partial \theta} = \sin\theta\frac{\partial}{\partial \theta}[r\cos\theta + \phi] = \sin\theta[-r\sin\theta] = -r\sin^2\theta$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}[r\cos\theta + \phi] = \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}[1] = 0$$

$$\therefore \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\cos\theta] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(-r\sin^2\theta)$$

$$= \frac{1}{r^2}2r\cos\theta + \frac{1}{r^2\sin\theta}[-r\,2\sin\theta\cos\theta] = \frac{2}{r}\cos\theta - \frac{2}{r}\cos\theta$$

$$= 0$$

So this field satisfies Laplace's equation.

Example 6.4 : *In a free space, $\rho_v = \frac{200\,\varepsilon_0}{r^{2.4}}$*

i) Use Poisson's equation, to find V as a function of r, if it is assumed that $r^2 E_r \to 0$ as $r \to 0$ and $V \to 0$ as $r \to \infty$. Use spherical co-ordinate system.

ii) Find potential V as a function of r using Gauss's law and line integral.

Solution : i) Poisson's equation states that,

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_0} \quad \text{... as free space } \varepsilon = \varepsilon_0$$

$$\therefore \qquad \nabla^2 V = -\frac{200\,\varepsilon_0}{r^{2.4}\,\varepsilon_0} = -\frac{200}{r^{2.4}}$$

From the conditions given it is clear that V is a function of *r* only *and* not the function of θ and ϕ.

$$\therefore \qquad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right]$$

$$\therefore \qquad \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = -\frac{200}{r^{2.4}}$$

$$\therefore \qquad \frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = -200\ r^{-0.4}$$

Integrate, $r^2 \frac{\partial V}{\partial r} = -\int 200\, r^{-0.4}\, dr + C_1$

$$\therefore \qquad r^2 \frac{\partial V}{\partial r} = -\frac{200\, r^{0.6}}{0.6} + C_1 = -333.33\, r^{0.6} + C_1 \qquad \text{... (1)}$$

As $\bar{E}$ is the function of *r* only we can write,

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}_r = E_r \bar{a}_r$$

$$\textit{and} \qquad E_r = -\frac{\partial V}{\partial r} \qquad \text{... (2)}$$

$$\therefore \qquad -r^2 E_r = -333.33\, r^{0.6} + C_1 \qquad \text{... (3)}$$

But as $r \to 0$, $r^2 E_r \to 0$ ... (given)

$$\therefore \qquad 0 = 0 + C_1$$

$$\therefore \qquad C_1 = 0 \qquad \text{... (4)}$$

Using in equation (1), $r^2 \frac{\partial V}{\partial r} = -333.33\, r^{0.6}$

$$\therefore \qquad \frac{\partial V}{\partial r} = -333.33\, r^{-1.4}$$

Integrate, $\qquad V = -333.33 \int r^{-1.4}\, dr + C_2$

$$= -333.33\,\frac{r^{-0.4}}{(-0.4)} + C_2 = \frac{833.325}{(r)^{0.4}} + C_2 \qquad \text{... (5)}$$

Use $V \to 0$ as $r \to \infty$

$$\therefore \quad 0 = \frac{833.325}{(\infty)^{0.4}} + C_2 = 0 + C_2$$

$$\therefore \quad C_2 = 0 \qquad \text{... (}$$

$$\therefore \quad \boxed{V = \frac{833.325}{(r)^{0.4}}\ V}$$

ii) Let us verify this using Gauss's law.

$$\nabla \cdot \bar{D} = \rho_v$$

$$\therefore \quad \nabla \cdot \varepsilon_0 \bar{E} = \rho_v$$

$$\therefore \quad \nabla \cdot \bar{E} = \frac{\rho_v}{\varepsilon_0} = \frac{+200\,\varepsilon_0}{r^{2.4}\,\varepsilon_0} = \frac{200}{r^{2.4}}$$

where $\bar{E} = E_r \bar{a}_r$ and no other component exis

Consider the radial component of $\bar{E}$ in spherical co-ordinate system and henc divergence of $\bar{E}$ is,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{200}{r^{2.4}}$$

$$\therefore \quad \frac{\partial}{\partial r}(r^2 E_r) = \frac{200}{r^{1.4}}$$

Integrate, $r^2 E_r = 200\frac{r^{-0.4+1}}{0.6} + C_1 = 333.33\, r^{0.6} + C_1$

But $r^2 E_r \to 0$ as $r \to 0$ hence $C_1 = 0$

$$\therefore \quad r^2 E_r = 333.33\, r^{0.6}$$

$$\therefore \quad \bar{E} = E_r \bar{a}_r = 333.33\, r^{-1.4}\,\bar{a}_r \quad V/m$$

Now $\quad V = -\int \bar{E} \cdot d\bar{L}$ where $d\bar{L} = dr\,\bar{a}_r$ in radial direction

$$\therefore \quad V = -\int 333.33\, r^{-1.4}\,\bar{a}_r \cdot dr\,\bar{a}_r = -333.33 \int r^{-1.4}\, dr$$

$$= -333.33\frac{r^{-0.4}}{-0.4} + C_2 = \frac{833.33}{(r)^{0.4}} + C_2$$

But $V = 0$ as $r \to \infty$ hence $C_2 = 0$

$$\therefore \quad \boxed{V = \frac{833.33}{(r)^{0.4}}\ V}$$

This is same as obtained above using Poisson's equation.

➡ **Example 6.6 :** *Solve the Laplace's equation for the potential field in the homogeneous region between the two concentric conducting spheres with radii a and b, such that b > a if potential V = 0 at r = b and V = V₀ at r = a. And find the capacitance between the two concentric spheres.* **[UPTU : 2003-04]**

Solution : The concentric conductors are shown in the Fig. 6.2.

At $r = b$, $V = 0$ hence the outer sphere is shown at zero potential.

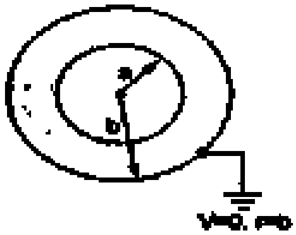

Fig. 6.2

The field intensity $\bar{E}$ will be only in radial direction hence V is changing only in radial direction as the radial distance r, and not the function of θ and ϕ.

According to Laplace's equation,

$$\nabla^2 V = 0$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \quad \text{... as V is function of r only}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

Integrate, $\quad r^2 \frac{\partial V}{\partial r} = \int 0 + C_1 = C_1 \qquad \text{... (1)}$

$$\therefore \quad \frac{\partial V}{\partial r} = \frac{C_1}{r^2} = C_1 r^{-2}$$

Integrate, $$V = \int C_1 r^{-2} dr + C_2 = \frac{C_1 r^{-1}}{-1} + C_2$$

$$\therefore \quad V = -\frac{C_1}{r} + C_2 \qquad \ldots (2)$$

Use the boundary conditions,

$V = 0$ at $r = b$ and $V = V_0$ at $r = a$

$$\therefore \quad 0 = -\frac{C_1}{b} + C_2 \quad \text{and} \quad V_0 = -\frac{C_1}{a} + C_2$$

Subtracting the two equations,

$$-V_0 = -\frac{C_1}{b} - \left(-\frac{C_1}{a}\right)$$

$$\therefore \quad -V_0 = C_1\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$\therefore \quad C_1 = \frac{-V_0}{\left[\frac{1}{a} - \frac{1}{b}\right]} = \frac{V_0}{\left[\frac{1}{b} - \frac{1}{a}\right]}$$

$$\therefore \quad C_2 = \frac{C_1}{b} = \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]}$$

$$\boxed{V = -\frac{V_0}{r\left[\frac{1}{b} - \frac{1}{a}\right]} + \frac{V_0}{b\left[\frac{1}{b} - \frac{1}{a}\right]} \ \text{V}}$$

This is the potential field in the region between the two spheres.

Now $$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}_r = -\frac{\partial}{\partial r}\left[\frac{-V_0}{r\left(\frac{1}{b} - \frac{1}{a}\right)}\right]\bar{a}_r \qquad \because C_2 \text{ is not function of } r$$

$$\boxed{\bar{E} = \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\bar{a}_r = \frac{-V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)r^2}\bar{a}_r \ \text{V/m}}$$

$$\therefore \quad \bar{D} = \varepsilon\bar{E} = \frac{-\varepsilon V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)r^2}\bar{a}_r = \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2} \ \text{C/m}^2$$

As per the boundary conditions between conductor and dielectric, the $\bar{D}$ is always normal to the surface hence $\bar{D}_N$.

$$\therefore \quad \rho_s = |\bar{D}_N| = |\bar{D}| = \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2} \ \text{C/m}^2$$

Now Q = Total charge on the surface of sphere of radius r

$$= \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2} \times \text{Surface area of sphere of radius } r$$

$$= \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2} \times 4\pi r^2 = \frac{4\pi\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} \ \text{C}$$

Now $C = \frac{Q}{V}$ where V = Potential between two spheres

$\therefore$ $V = V_0$ = Potential difference between two spheres

$$\therefore \quad \boxed{C = \frac{Q}{V_0} = \frac{\frac{4\pi\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}}{V_0} = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} \ \text{F}}$$

This is the capacitance of a spherical capacitor.

➡ **Example 6.8 :** *Use Laplace's equation to find the capacitance per unit length of a co-axial cable of inner radius 'a' m and outer radius 'b' m. Assume $V = V_0$ at $r = a$ and $V = 0$ at $r = b$.*

Solution : The co-axial cable is shown in the Fig. 6.3.

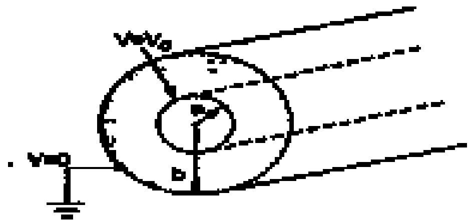

Fig. 6.3

Consider cylindrical co-ordinate system. The field intensity $\bar{E}$ is in radial direction from inner to outer cylinder hence V is a function of r only and not the function of ϕ and z.

Using Laplace's equation,

$$\nabla^2 V = 0$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \qquad \ldots V = f(r) \text{ only}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

Integrate,

$$r\frac{\partial V}{\partial r} = \int 0 + C_1 = C_1 \qquad \ldots (1)$$

$$\therefore \quad \frac{\partial V}{\partial r} = \frac{C_1}{r}$$

Integrate,

$$V = \int \frac{C_1}{r} + C_2 = C_1 [ln\, r] + C_2 \qquad \ldots (2)$$

Using boundary conditions, $V = 0$ at $r = b$ and $V = V_0$ at $r = a$,

$$0 = C_1\, ln(b) + C_2 \quad \text{and} \quad V_0 = C_1\, ln(a) + C_2$$

Subtracting,

$$-V_0 = C_1 \{ln(b) - ln(a)\} = C_1 \left\{ln\left(\frac{b}{a}\right)\right\}$$

$$\therefore \quad C_1 = \frac{-V_0}{ln\left(\frac{b}{a}\right)} = \frac{V_0}{ln\left(\frac{a}{b}\right)}$$

and

$$C_2 = -C_1\, ln(b) = \frac{-V_0\, ln(b)}{ln\left(\frac{a}{b}\right)}$$

$$\boxed{V = \frac{V_0}{ln\left(\frac{a}{b}\right)}\, ln(r) - \frac{V_0\, ln(b)}{ln\left(\frac{a}{b}\right)} \text{ V}}$$

Now

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}_r$$

$$= -\frac{\partial}{\partial r}\left[\frac{V_0\, ln(r)}{ln\left(\frac{a}{b}\right)}\right]\bar{a}_r \qquad \ldots C_2 \text{ is not function of } r$$

$$\boxed{\bar{E} = -\frac{V_0}{ln\left(\frac{a}{b}\right)}\left[\frac{\partial}{\partial r} ln(r)\right]\bar{a}_r = -\frac{V_0}{r\, ln\left(\frac{a}{b}\right)}\bar{a}_r \quad \text{V/m}}$$

$\therefore$ $$D = \varepsilon E = \frac{-V_0 \varepsilon}{r\, ln\left(\frac{a}{b}\right)} \bar{a}_r = \frac{V_0 \varepsilon}{r\, ln\left(\frac{b}{a}\right)} \bar{a}_r \text{ C/m}^2$$

Now $\bar{D}$ is existing normal to the surface as per the boundary conditions.

$\therefore$ $$\bar{D} = \bar{D}_N = \frac{V_0 \varepsilon}{r\, ln\left(\frac{b}{a}\right)} \bar{a}_r$$

$\therefore$ $$\rho_S = |\bar{D}_N| = \frac{V_0 \varepsilon}{r\, ln\left(\frac{b}{a}\right)} \text{ C/m}^2$$

ρ_S exists on entire surface area of inner cylinder.

$\therefore$ $$Q = \rho_S \times \text{Surface area of inner cylinder}$$

$$= \frac{V_0 \varepsilon}{r\, ln\left(\frac{b}{a}\right)} \times 2\pi r \times L = \frac{V_0 \varepsilon 2\pi L}{ln\left(\frac{b}{a}\right)} \text{ C}$$

The potential difference between the two cylinders is V_0. Thus $V = V_0$.

$\therefore$ $$C = \frac{Q}{V} = \frac{\frac{V_0 \varepsilon 2\pi L}{ln\left(\frac{b}{a}\right)}}{V_0}$$

$\therefore$ $$C = \frac{2\pi\varepsilon L}{ln\left(\frac{b}{a}\right)} \text{ F}$$

The capacitance per unit length i.e. L = 1 m is,

$$\boxed{C = \frac{2\pi\varepsilon}{ln\left(\frac{b}{a}\right)} \text{ F/m}}$$

➡ **Example 6.12 :** *Given the volume charge density $\rho_v = -2 \times 10^7 \varepsilon_0 \sqrt{x}$ C/m^3 in free space, let $V = 0$ at $x = 0$ and $V = 2$ V at $x = 2.5$ mm. Find V at $x = 1$ mm.*

Solution : As $\rho_v \neq 0$, use Poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = -\frac{[-2 \times 10^7 \varepsilon_0 \sqrt{x}]}{\varepsilon_0} \qquad \ldots \varepsilon = \varepsilon_0 \text{ as free space}$$

Now V is a function of x alone, hence $\nabla^2 V = \frac{\partial^2 V}{\partial x^2}$.

$\therefore$ $$\frac{\partial^2 V}{\partial x^2} = 2 \times 10^7 x^{1/2}$$

Integrating, $$\frac{\partial V}{\partial x} = \frac{2 \times 10^7 x^{3/2}}{\frac{3}{2}} + C_1 = 13.33 \times 10^6 x^{1.5} + C_1$$

Integrating, $$V = \int [13.33 \times 10^6 x^{1.5} + C_1]\, dx + C_2$$

$$= \frac{[13.33 \times 10^6 x^{2.5}]}{2.5} + C_1 x + C_2$$

$\therefore$ $$V = 5.33 \times 10^6 x^{2.5} + C_1 x + C_2$$

As $x = 0$, $V = 0$ hence $0 = 0 + 0 + C_2$, $C_2 = 0$

At $x = 2.5$ mm, $V = 2$ V hence

$$2 = 5.33 \times 10^6 (2.5 \times 10^{-3})^{2.5} + C_1 (2.5 \times 10^{-3})$$

$\therefore$ $$C_1 = 133.75$$

$\therefore$ $$V = 5.33 \times 10^6 x^{2.5} + 133.75 x \text{ V}$$

At $x = 1$ mm $= 1 \times 10^{-3}$ m,

$$V = 5.33 \times 10^6 (1 \times 10^{-3})^{2.5} + 133.75 (1 \times 10^{-3})$$

$$= 0.30229 \text{ V}$$