ELECTRO MAGNETIC FIELDS (3-0-0)
LECTURE NOTES
B. TECH
(II YEAR – III SEM)

Prepared by:
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Electromagnetic Fields (3-0-0)

Prerequisites:

1. Mathematics-I
2. Mathematics-II

Course Outcomes

At the end of the course, students will demonstrate the ability

1. To understand the basic laws of electromagnetism.
2. To obtain the electric and magnetic fields for simple configurations under static conditions.
3. To analyse time-varying electric and magnetic fields.
4. To understand Maxwell's equation in different forms and different media.
5. To understand the propagation of EM waves.

Module 1: (08 Hours)

Co-ordinate systems & Transformation: Cartesian co-ordinates, circular cylindrical coordinates, spherical coordinates. Vector Calculus: Differential length, Area & Volume, Line, surface and volume Integrals, Del operator, Gradient of a scalar, Divergence of a vector & Divergence theorem, Curl of a vector & Stoke's theorem, Laplacian of a scalar.

Module 2: (10 Hours)

Electrostatic Fields: Coulomb's Law, Electric Field Intensity, Electric Fields due to a point, line, surface and volume charge, Electric Flux Density, Gauss's Law- Maxwell's Equation, Applications of Gauss's Law, Electric Potential, Relationship between E and V- Maxwell's Equation and Electric Dipole & Flux Lines, Energy Density in Electrostatic Fields., Current and current density, Ohms Law in Point form, Continuity of current, Boundary conditions. Electrostatic boundary-value problems: Poisson's and Laplace's Equations, Uniqueness Theorem, General procedures for solving Poisson's and Laplace's equations, Capacitance.

Module 3: (06 Hours)

Magneto static Fields: Magnetic Field Intensity, Biot-Savart's Law, Ampere's circuit Law-Maxwell Equation, applications of Ampere's law, Magnetic Flux Density-Maxwell's equations. Maxwell's equation for static fields, Magnetic Scalar and Vector potentials. Magnetic Boundary Conditions.

Module 4: (10 Hours)

Electromagnetic Field and Wave propagation: Faraday's Law, Transformer & Motional Electromagnetic Forces, Displacement Current, Maxwell's Equation in Final forms, Time-Harmonic Field. Electromagnetic Wave Propagation: Wave Propagation in lossy Dielectrics, Plane Waves in loss less Dielectrics, Free space, Good conductors Power & Poynting vector.

TEXTBOOKS:

1. Matthew N. O. Sadiku, Principles of Electromagnetics, 6th Ed., Oxford Intl. Student Edition, 2014.

REFERENCE BOOKS:

1. C. R. Paul, K. W. Whites, S. A. Nasor, Introduction to Electromagnetic Fields, 3rd Ed, TMH.
2. W.H. Hyat, Electromagnetic Field Theory, 7th Ed, TMH.
3. A. Pramanik, "Electromagnetism - Theory and applications", PHI Learning Pvt. Ltd, New Delhi, 2009.
4. A. Pramanik, "Electromagnetism-Problems with solution", Prentice Hall India, 2012.
5. G.W. Carter, "The electromagnetic field in its engineering aspects", Longmans, 1954.
6. W.J. Duffin, "Electricity and Magnetism", McGraw Hill Publication, 1980.
7. W.J. Duffin, "Advanced Electricity and Magnetism", McGraw Hill, 1968.
8. E.G. Cullwick, "The Fundamentals of Electromagnetism", Cambridge University Press, 1966.
9. B. D. Popovic, "Introductory Engineering Electromagnetics", Addison- Wesley Educational Publishers, International Edition, 1971.
10. W. Hayt, "Engineering Electromagnetics", McGraw Hill Education, 2012.

MODULE-I

1. Co-ordinate System and Transformation
   - Cartesian Co-ordinates
   - Circular Cylindrical Co-ordinates
   - Spherical Co-ordinates

2. Vector Calculus
   - Differential length, Area & volume
   - Line, surface and volume Integrals
   - Del operator
   - Gradient of a scalar
   - Divergence of a vector & Divergence theorem
   - Curl of a vector & Stoke's theorem
   - Laplacian of a scalar

Chapter-1

Co-ordinate System and Transformation

1.1 Introduction

Electromagnetic is a branch of physics or electrical engineering which is used to study the electric and magnetic phenomena. The electric and magnetic fields are closely related to each other.

Field is a function that specifies a quantity everywhere in a region or space. If the field produced is due to magnetic effects, it is called **magnetic field**. Similarly, the field produced by an electric charge is called an **electric field**. Moving charges produce a current and current-carrying conductor produces a magnetic field. In such a case, electric and magnetic fields are related to each other. Such a field is called the **electromagnetic field**.

Various applications of the electromagnetic field are:

- Electrical machines
- Radiofrequency communication
- Microwave engineering
- Sattelite communication
- Radar technology
- Antenna technology
- Nuclear research
- Fiber optics
- Elctromagnetic Interference etc.

The vector analysis is a mathematical shorthand tool with which electromagnetic concepts can be most conviniently expressed.

1.2 Scalar and Vector

Various quantities involved in the study of electromagnetics can be classified as

1. Scalar and 2. Vector

1.2.1 Scalar

A scalar is a quantity which is characterized by its magnitude. The various examples of scalar quantity are temperature, mass, volume, density, speed, elctric charge etc.

1.2.2 Vector

A vector is a quantity that is characterized by both its magnitude and direction. The various examples of vector quantity are force, electricity, displacement, electric field intensity, magnetic field intensity, acceleration, etc.

1.3 Co-ordinate System

The physical quantities, we shall be dealing with in the electromagnetic field are functions of space and time. To describe the spatial variations of these quantities, require an appropriate coordinate system which may be orthogonal or non-orthogonal.

An orthogonal system is one in which the co-ordinates are mutually perpendicular. For example

- Cartesian or Rectangular Co-ordinate System
- Circular Cylindrical Co-ordinate System
- Spherical Co-ordinate System etc.

A hard problem in one co-ordinate system may turn out to be easy in another system. Hence we will study transformation of co-ordinate system.

Non-orthogonal systems are hard to work with and they are of little or no practical use. Let us discuss orthogonal systems in detail.

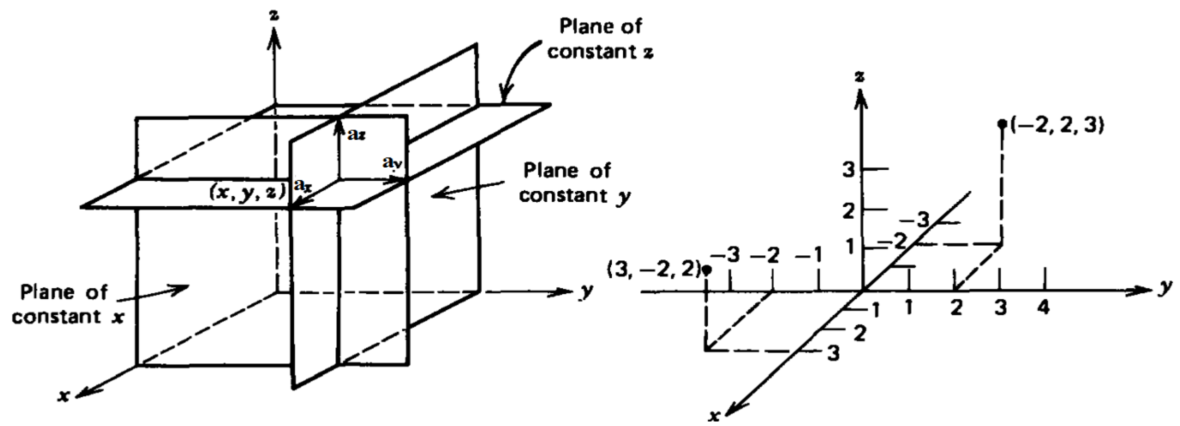
1.4 Rectangular or Cartesian Co-ordinate System

The most common and often preferred coordinate system is defined by the intersection of three mutually perpendicular planes as shown in Figure 1-4a. Lines parallel to the lines of intersection between planes define the coordinate axes (x, y, z), where the x axis lies perpendicular to the plane of constant x or yz-plane, the y axis is perpendicular to the plane of constant y or xz-plane, and the z axis is perpendicular to the plane of constant z or xy-plane. Once an origin is selected with coordinate (0, 0, 0), any other point in the plane is found by specifying its x-directed, y-directed, and z-directed distances from this origin as shown for the coordinate points located in Figure 1-4b.

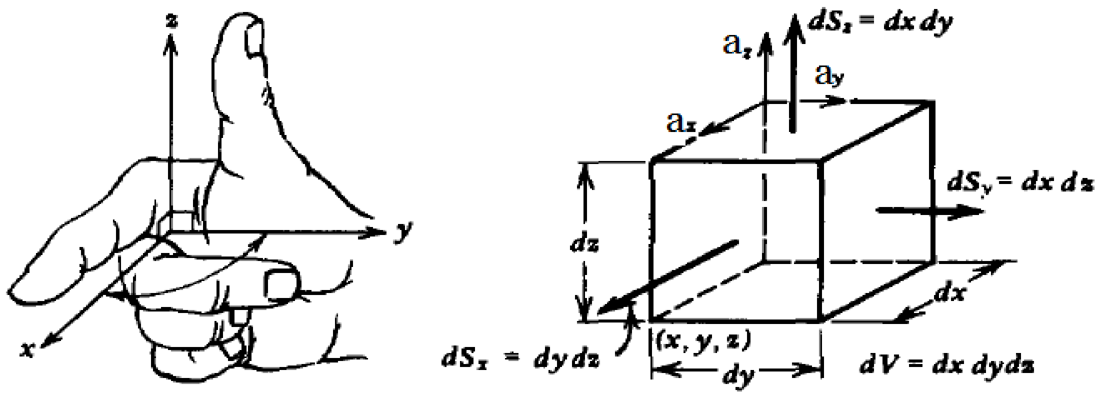
By convention, a right-handed coordinate system is always used whereby one curls the fingers of his or her right hand in the direction from x to y so that the forefinger is in the x direction and the middle finger is in the y direction. The thumb then points in the z direction. This convention is necessary to remove directional ambiguities in theorems to be derived later.

Coordinate directions are represented by unit vectors a_x, a_y, and a_z, each of which has a unit length and points in the direction along one of the coordinate axes. Rectangular coordinates are often the simplest to use because the unit vectors always point in the same direction and do not change direction from point to point.

A rectangular differential volume is formed when one moves from a point *(x, y,* z) by an incremental distance *dx, dy,* and *dz* in each of the three coordinate directions as shown in Figure 1-4c. To distinguish surface elements, we subscript the area element of each face with the coordinate perpendicular to the surface.



(a) (b)



(c)

1.4.1 Position and Distance Vector

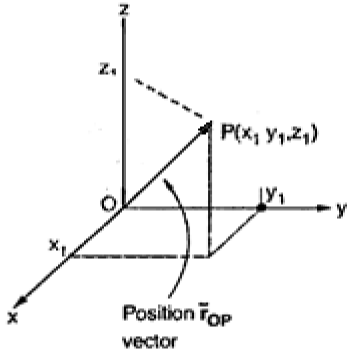


Fig. 1.4.1 Position vector

Consider a point P (x_1, y_1, z_1) in the Cartesian coordinate system. Then the **position vector** of point P is the distance of point P from the origin, directed from origin to point P. This is also called the **radius vector** as shown in figure 1.4.1.

Thus the position vector of point P can be represented as,

$$\overrightarrow{r_{OP}} = x_1\overrightarrow{a_x} + y_1\overrightarrow{a_y} + z_1\overrightarrow{a_z} \;\ldots\ldots (1)$$

The magnitude of this vector in terms of three mutually perpendicular components is given by

$$|\overrightarrow{r_{OP}}| = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2} \ldots\ldots. (2)$$

Thus if point P has co-ordinates (1, 2, 3) then its position vector is,

$$\overrightarrow{r_{OP}} = 1\overrightarrow{a_x} + 2\overrightarrow{a_y} + 3\overrightarrow{a_z} \quad \text{and}$$

$$|\overrightarrow{r_{OP}}| = \sqrt{(1)^2 + (2)^2 + (3)^2} = 3.7416$$

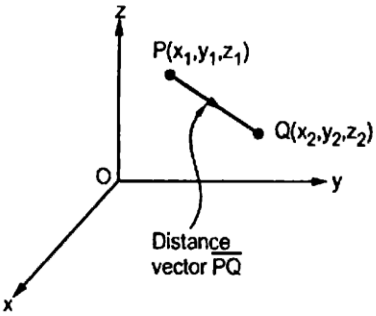


Fig. 1.4.2 Distance vector

Now consider the two points in a Cartesian coordinate system, P and Q with the coordinates (x1, y1, z1) and (x2, y2, z2) respectively. The points are shown in Fig. 1.4.2. The individual position vectors of the points are,

$$\vec{P} = x_1\overrightarrow{a_x} + y_1\overrightarrow{a_y} + z_1\overrightarrow{a_z}$$

$$\vec{Q} = x_2\overrightarrow{a_x} + y_2\overrightarrow{a_y} + z_2\overrightarrow{a_z}$$

Then the distance or the displacement from P to Q is represented by a **distance vector** $\overrightarrow{PQ}$ and is given by,

$$\overrightarrow{PQ} = \vec{Q} - \vec{P} = (x_2 - x_1)\overrightarrow{a_x} + (y_2 - y_1)\overrightarrow{a_y} + (z_2 - z_1)\overrightarrow{a_z} \ldots.. (3)$$

This is also called **separation vector**.

The magnitude of this vector is given by,

$$\left|\overrightarrow{PQ}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \ldots\ldots (4)$$

Unit vector along the direction PQ as

$$\overrightarrow{\mathrm{a_{PQ}}} = \text{unit vector along PQ} = \frac{\overrightarrow{\mathrm{PQ}}}{|\overrightarrow{\mathrm{PQ}}|} \ldots\ldots. (5)$$

Example 1.3 : *Given three points in cartesian co-ordinate system as* $A(3,-2,1)$, $B(-3,-3,5)$, $C(2,6,-4)$.

Find : i) The vector from A to C.

ii) The unit vector from B to A.

iii) The distance from B to C.

iv) The vector from A to the midpoint of the straight line joining B to C.

Solution : The position vectors for the given points are,

$\bar{A} = 3\bar{a}_x - 2\bar{a}_y + \bar{a}_z$, $\bar{B} = -3\bar{a}_x - 3\bar{a}_y + 5\bar{a}_z$, $\bar{C} = 2\bar{a}_x + 6\bar{a}_y - 4\bar{a}_z$

i) The vector from A to C is,

$$\overline{AC} = \bar{C} - \bar{A} = [2-3]\bar{a}_x + [6-(-2)]\bar{a}_y + [-4-1]\bar{a}_z$$
$$= -\bar{a}_x + 8\bar{a}_y - 5\bar{a}_z$$

ii) For unit vector from B to A, obtain distance vector $\overline{BA}$ first.

$\therefore$ $\overline{BA} = \bar{A} - \bar{B}$ ... as starting is B and terminating is A

$$= [3-(-3)]a_x + [(-2)-(-3)]\bar{a}_y + [1-5]\bar{a}_z$$
$$= 6\bar{a}_x + \bar{a}_y - 4\bar{a}_z$$

$\therefore$ $$|\overline{BA}| = \sqrt{(6)^2 + (1)^2 + (-4)^2} = 7.2801$$

$$\therefore \bar{a}_{BA} = \frac{\overline{BA}}{|\overline{BA}|} = \frac{6\bar{a}_x + \bar{a}_y - 4\bar{a}_z}{7.2801} = \mathbf{0.8241}\ \bar{a}_x + \mathbf{0.1373}\ \bar{a}_y - \mathbf{0.5494}\ \bar{a}_z$$

iii) For distance between B and C, obtain $\overline{BC}$

$$\overline{BC} = \bar{C} - \bar{B} = [2-(-3)]\bar{a}_x + [6-(-3)]\bar{a}_y + [(-4)-(5)]\bar{a}_z = 5\bar{a}_x + 9\bar{a}_y - 9\bar{a}_z$$

$\therefore$ $$\text{Distance BC} = \sqrt{(5)^2 + (9)^2 + (-9)^2} = \mathbf{13.6747}$$

iv) Let $B(x_1, y_1, z_1)$ and $C(x_2, y_2, z_2)$ then the co-ordinates of midpoint of BC are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

$$\therefore \text{Midpoint of BC} = \left(\frac{-3+2}{2}, \frac{-3+6}{2}, \frac{5-4}{2}\right) = (-0.5, 1.5, 0.5)$$

Hence the vector from A to this midpoint is

$$= [-0.5-3]\bar{a}_x + [1.5-(-2)]\bar{a}_y + [0.5-1]\bar{a}_z = -\mathbf{3.5}\ \bar{a}_x + \mathbf{3.5}\ \bar{a}_y - \mathbf{0.5}\ \bar{a}_z$$

1.5 Circular Cylindrical Co-ordinate System

The circular cylindrical co-ordinate system is the three dimensional polar co-ordinate system. The surfaces used to define the cylindrical co-ordinate system are

1. Plane of constant z which is parallel to xy plane.
2. A cylinder of radius ρ with z-axis as the axis of the cylinder.
3. A half-plane perpendicular to xy-plane and at an angle ϕ with respect to xz-plane. The angle ϕ is called the **azimuthal angle**.

The range of the variables are,

$$0 \le \rho < \infty \quad \ldots\ldots (1)$$
$$0 \le \emptyset < 2\pi \quad \ldots\ldots (2)$$
$$-\infty < z < \infty \ldots\ldots (3)$$

The point P in cylindrical co-ordinate system has three co-ordinates ρ, ϕ, and z whose values lie in the respective ranges given by the equations (1), (2), and (3). The point P (ρ, ϕ, z) can be shown as in Fig. 1.5.1

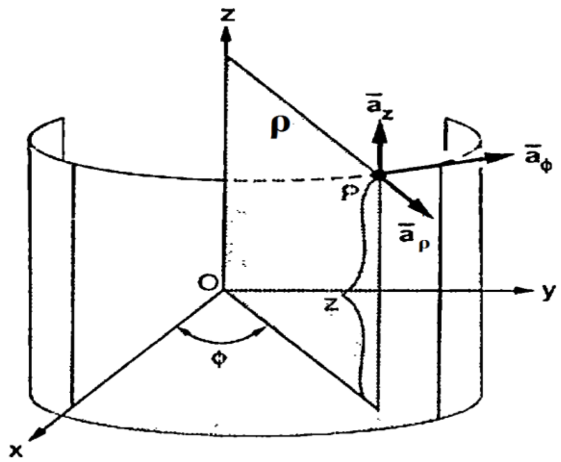


Fig.1.5.1 Cylindrical Co-ordinate System

Key points: Note that angle ϕ is expressed in radians and for ϕ, anticlockwise measurement is treated positive while clockwise measurement is treated negative.

The point P can be defined as the intersection of three surfaces in cylindrical co-ordinate system. These three surfaces are,

ρ = Constant which is a circular cylinder with z-axis as its axis

ϕ = Constant plane which is a vertical plane perpendicular to xy plane making angle ϕ with respect to xz-plane

z = Constant plane is a plane parallel to xy plane.

These surfaces are shown in Fig.1.5.2

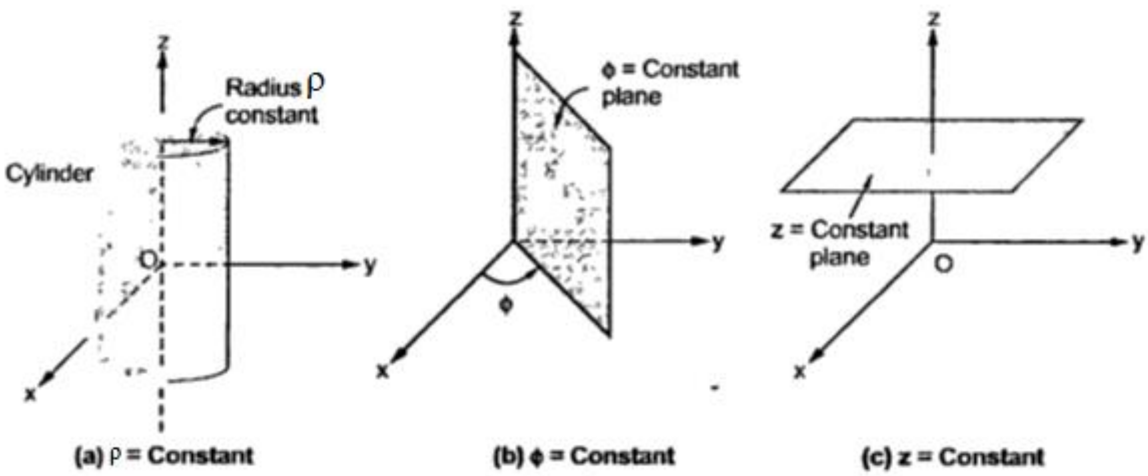


Fig. 1.5.2

Similar to the Cartesian coordinate system, there are three unit vectors in the ρ, ϕ and z directions denoted as $\overrightarrow{a_\rho}$, $\overrightarrow{a_\phi}$ and $\overrightarrow{a_z}$. These are mutually perpendicular to each other.

The $\overrightarrow{a_\rho}$ lies in a plane parallel to the xy-plane and is perpendicular to the surface of the cylinder at a given point, coming radially outward.

The unit vector $\overrightarrow{a_\phi}$ lies also in a plane parallel to the xy-plane but it is tangent to the cylinder and pointing in a direction of increasing ϕ, at the given point.

The unit vector $\overrightarrow{a_z}$ is parallel to the z-axis and directed towards increasing z.

Hence vector of point P in Fig. 1.5.1 can be represented as,

$$\vec{P} = P_\rho\overrightarrow{a_\rho} + P_\phi\overrightarrow{a_\phi} + P_z\overrightarrow{a_z} \text{ .... (4)}$$

Where P_ρ is radius, P_ϕ is angle ϕ and P_z is z-co-ordinate of point P in the cylinder.

1.5.1 Relationship between Cartesian and Cylindrical Coordinate System

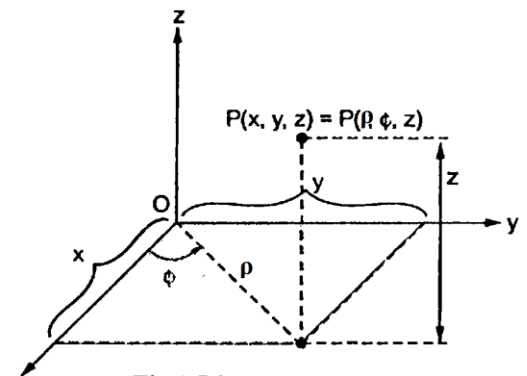


Fig.1.5.3

Consider a point P whose Cartesian coordinates are x, y, z while the cylindrical co-ordinates are ρ, ϕ, and z, as shown in Fig. 1.5.3.

Looking at the xy-plane the transformation from cylindrical to cartesian co-ordinate can be obtained from the equations,

$$x = \rho\cos\emptyset,\ y = \rho\sin\emptyset, z = z \text{ ...... (5)}$$

Thus transformation from cartesian to cylindrical can be obtained from the equations,

$$\rho = \sqrt{(x^2 + y^2)}, \emptyset = \tan^{-1}\frac{y}{x}, z = z \text{ .... (6)}$$

Key Points: While using equations (6) note that ρ is positive or zero, hence the positive sign of square root must be considered. While calculating ϕ make sure the signs of x and y. If both are positive, ϕ is positive in the first quadrant. If x is negative and y is positive then the point is in the second quadrant, hence ϕ must be within $+90^0$ and $+180^0$ i.e. within -180^0 and 270^0. Thus for x = -2 and y =1 we get $\emptyset = \tan^{-1}\left[\frac{1}{-2}\right] = -26.56^0$ but it should be taken as -26.56^0+180^0 i.e. 154.43^0. Hence when x is negative it is necessary to add 180^0 to the ϕ calculated using the $\tan^{-1}$ function, to obtain accurate ϕ corresponding to the point.

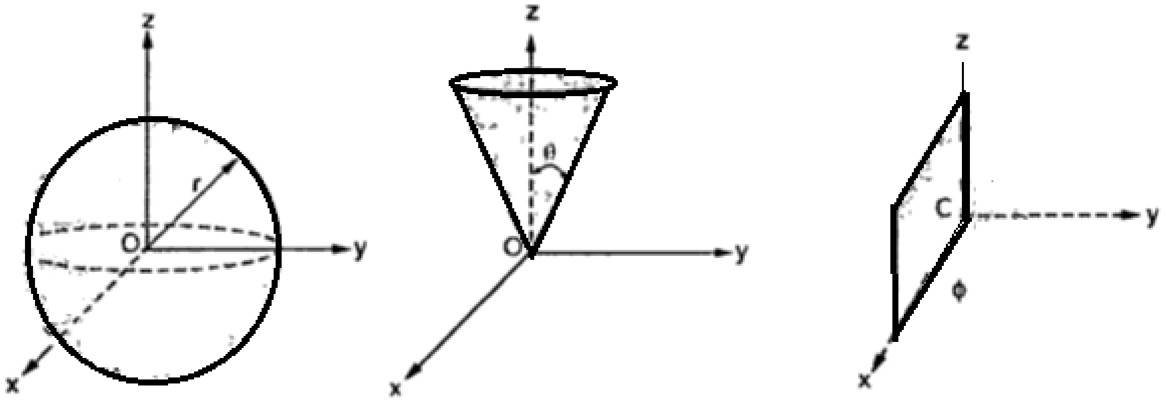
When y is negative and x is positive then ϕ is in fourth quadrant i.e. within 0^0 and -90^0 i.e. 270^0 and 360^0. Similarly, when x and y, both are negative, the point is in the third quadrant, and accordingly, ϕ must be between -90^0 to -180^0 i.e. $+180^0$ and $+270^0$. So 180^0 must be subtracted from the ϕ calculated by tan-1 function, to get accurate ϕ when both x and y are negative. Thus if x = y = -3 then ϕ = tan-1[-3/-3] = 45^0 but actually it is $45^0 – 180^0 = -135^0$ i.e. $-135^0+360^0 = +225^0$.

1.6 Spherical Coordinate System

The surfaces which are used to define the spherical coordinate system on the three Cartesian axes are,

1. Sphere of radius r, origin as the center of the sphere.
2. A right circular cone with its apex at the origin and its axis as the z-axis. Its half-angle is θ and called a **colatitude angle**. It rotates about the z-axis and θ varies from 0 to 180^0.
3. A half-plane perpendicular to xy plane containing z-axis, making an angle ϕ with xz plane.

Thus the three coordinates of a point P in the spherical coordinate system are (r, θ, ϕ). The surfaces are shown in the Fig.1.6.1.



(a) Sphere of radius r with centre as origin

(b) Right circular cone with apex at origin

(c) Half plane perpendicular to xy plane

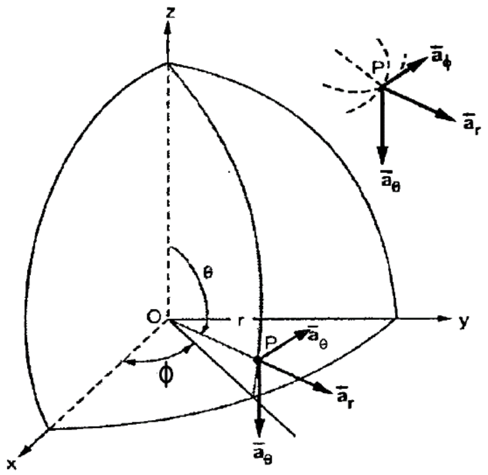
Fig. 1.6.1

The ranges of the variables are,

$$0 \le r < \infty \; \ldots\ldots (1)$$

$$0 \le \theta \le \pi \; \ldots.. (2)$$

$$0 \le \emptyset < 2\pi \; \ldots\ldots (3)$$



Similar to other two co-ordinate systems, there are three unit vectors in the r, θ and ϕ directions denoted as $\overrightarrow{a_r}, \overrightarrow{a_\theta}$ and $\overrightarrow{a_\phi}$. These unit vectors are mutually perpendicular to each other and are shown in Fig. 1.6.2.

The unit vector a_r is directed from the centre of the sphere which is the origin to the given point P. It is directed radially outward, normal to the sphere. It lies in the cone θ = constant and plane ϕ = constant.

Fig.1.6.2 Spherical Co-ordinate System

The unit vector $\overrightarrow{a_\theta}$ is tangent to the sphere and oriented in the direction of increasing θ. It is normal to the conical surface.

The third unit vector $\overrightarrow{a_\phi}$ is tangent to the sphere and also tangent to the conical surface. It is oriented in the direction of increasing ϕ and same as defined in the cylindrical co-ordinate system.

Hence vector of point P can be represented as

$$\vec{P} = P_r\overrightarrow{a_r} + P_\theta\overrightarrow{a_\theta} + P_\phi\overrightarrow{a_\phi} \; ..... (4)$$

Where P_r is the radius r and P_θ and P_ϕ are the two angle components of point P.

1.6.1 Relationship between Cartesian and Spherical Co-ordinate System

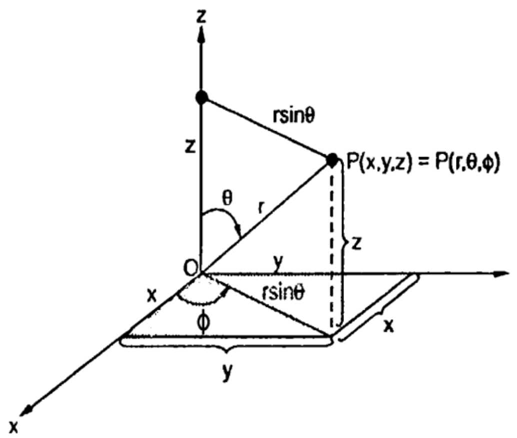


Fig. 1.6.3 Relationship between cartesian and spherical systems

Consider a point P whose Cartesian co-ordinates are x, y, z while the spherical co-ordinates are r, θ and ϕ, as shown in Fig. 1.6.3.

Looking at the xy-plane, the transformation from spherical to Cartesian can be obtained from the equations,

$$x = r\sin\theta\cos\emptyset\ ,\ y = r\sin\theta\sin\emptyset \text{ and } z = r\cos\theta \ \ldots\ldots (5)$$

Now r can be expressed as,

$$x^2 + y^2 + z^2 = r^2sin^2\theta cos^2\emptyset + r^2sin^2\theta sin^2\emptyset + r^2cos^2\theta$$

$$= r^2sin^2\theta[cos^2\emptyset + sin^2\emptyset] + r^2cos^2\theta$$

$$= r^2[sin^2\theta + cos^2\theta] = r^2$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

While $\tan\phi = \frac{y}{x}$ and $\cos\theta = \frac{z}{r}$

As r is known, θ can be obtained.

Thus the transformation from cartesian to spherical co-ordinate system can be obtained from the equations,

$$r = \sqrt{x^2 + y^2 + z^2}, \theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] \ and\ \emptyset = \tan^{-1}\frac{y}{x} \ \ldots\ldots. (6)$$

Key Points: While using above formulae, care must be taken to place the angle θ and ϕ in correct quadrants according to the signs of x, y and z.

1.7 Vector Multiplication

Consider two vectors $\vec{A}$ and $\vec{B}$. There are two types of products existing depending on the result of the multiplication. These two types of products are,

1. Scalar or Dot product
2. Vector or Cross product

Let us discuss the characteristics of these two products.

1.7.1 Scalar or Dot product

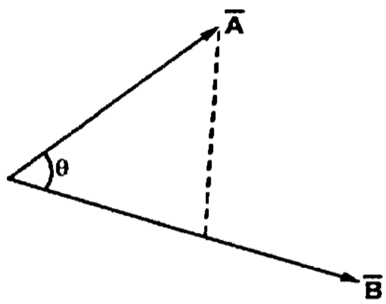


Fig. 1.7.1

The scalar or dot product of two vectors $\vec{A}$ and $\vec{B}$ is denoted as $\vec{A}\ .\ \vec{B}$ and defined as the product of the magnitude of A and magnitude of B and the cosine of the smaller angle between them.

$$\vec{A}\ .\ \vec{B} = |\vec{A}||\vec{B}|\cos\theta_{AB} \ldots\ldots. (1)$$

The result of dot product is scalar hence it is also called as scalar product.

1.7.2 Properties of Scalar or Dot product

1. If the two vectors are parallel to each other i.e. $\theta = 0^0$ then $\cos\theta_{AB} = 1$ thus

   $\vec{A}\ .\ \vec{B} = |\vec{A}||\vec{B}| \ldots\ldots (2)$

2. If the two vectors are perpendicular to each other i.e. $\theta = 90^0$ then $\cos\theta_{AB} = 0$ thus $\vec{A}\ .\ \vec{B} = 0 \ \ldots\ldots. (3)$

3. The dot product obeys commutative law, $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ …… (4)
4. The dot product obeys distributive law, $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$ …. (5)
5. If the dot product of a vector with itself is performed, the result is square of the magnitude of that vector.

$$\vec{A} \cdot \vec{A} = |\vec{A}||\vec{A}|\cos 0^0 = |\vec{A}|^2 \; \ldots \ldots (6)$$

6. Dot product of different unit vectors is zero.

$$\overrightarrow{a_x}.\overrightarrow{a_y} = \overrightarrow{a_y}.\overrightarrow{a_x} = \overrightarrow{a_x}.\overrightarrow{a_z} = \overrightarrow{a_z}.\overrightarrow{a_x} = \overrightarrow{a_y}.\overrightarrow{a_z} = \overrightarrow{a_z}.\overrightarrow{a_y} = o \; \ldots\ldots (7)$$

7. Any unit vector dotted with itself is unity

$$\overrightarrow{a_x}.\overrightarrow{a_x} = \overrightarrow{a_y}.\overrightarrow{a_y} = \overrightarrow{a_z}.\overrightarrow{a_z} = 1 \ldots\ldots (8)$$

8. Consider two vectors in cartesian co-ordinate system,

$$\vec{A} \cdot \vec{B} = (A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z}).(B_x\overrightarrow{a_x} + B_y\overrightarrow{a_y} + B_z\overrightarrow{a_z})$$
$$= A_x.B_x + A_y.B_y + A_z.B_z \ldots\ldots\ldots\ldots (9)$$

1.7.3 Applications of Scalar or Dot product

The applications of dot product are,

1. **Two determine the angle between two vectors.**
   The angle can be determined as,

$$\theta = \cos^{-1}\left\{\frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}\right\} \ldots\ldots\ldots\ldots (10)$$

2. **To find the component of a vector in a given direction.**

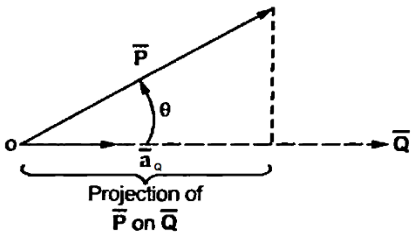


Fig. 1.7.2

The **scalar projection** of P on Q is given by

$$P_Q = |\vec{P}|\cos\theta = |\vec{P}|\frac{\vec{P}.\vec{Q}}{|\vec{P}||\vec{Q}|} = \vec{P}.\frac{\vec{Q}}{|\vec{Q}|} = \vec{P}.\overrightarrow{a_Q} \ldots.. (11)$$

$$\text{As } \overrightarrow{a_Q} = \frac{\vec{Q}}{|\vec{Q}|}$$

The **vector projection** of P on Q is given by

$$\overrightarrow{P_Q} = P_Q\overrightarrow{a_Q} = P_Q\frac{\vec{Q}}{|\vec{Q}|} \ldots\ldots. (12)$$

Example 1.10 : *Given vector field* $\overline{G} = (y-1)\bar{a}_x + 2x\bar{a}_y$. *Find this vector field at* $P(2,3,1)$ *and its projection on* $\overline{B} = 5\bar{a}_x - \bar{a}_y + 2\bar{a}_z$.

Solution : The field $\overline{G}$ at point P is,

$\overline{G}$ at P $= 2\bar{a}_x + 4\bar{a}_y$ ... Substituting co-ordinates of P in $\overline{G}$

To find its projection on $\overline{B}$, first find $\bar{a}_B$, the unit vector in the direction of $\overline{B}$.

$$\therefore \quad \bar{a}_B = \frac{\overline{B}}{|\overline{B}|} = \frac{5\bar{a}_x - \bar{a}_y + 2\bar{a}_z}{\sqrt{(5)^2 + (-1)^2 + (2)^2}}$$

$$= 0.9128\,\bar{a}_x - 0.1825\,\bar{a}_y + 0.3651\,\bar{a}_z$$

Hence projection of $\overline{G}$ at P on the vector $\overline{B}$ is,

$$= (\overline{G} \text{ at P}) \bullet \bar{a}_B$$

$$= (2 \times 0.9128) + (4 \times -0.1825) + (0 \times 0.3651) = \mathbf{1.0956}$$

1.7.4 Vector or Cross product

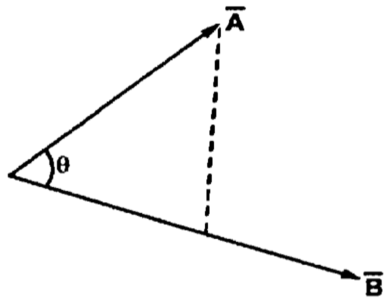


Fig. 1.7.1

Cross product of two vectors $\vec{A}$ and $\vec{B}$ is denoted as $\vec{A}$ x $\vec{B}$ and defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the smaller angle between $\vec{A}$ and $\vec{B}$. But this product is a vector quantity ad has direction perpendicular to the plane containing the two vectors $\vec{A}$ and $\vec{B}$. The direction of the cross product is along the perpendicular direction to the plane which is in the direction of advancement of right hand screw when $\vec{A}$ is turned into $\vec{B}$.

Mathematically the cross product is expressed as,

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta_{AB}\overrightarrow{a_N} \text{ …….. (13)}$$

Where $\overrightarrow{a_N}$ = Unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ in the direction decided by the right hand screw rule.

1.7.5 Properties of Vector or Cross product

The various properties of cross product are,

1. The commutative law is not applicable.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A} \text{ …… (14)}$$

But $\vec{B} \times \vec{A} = -[\vec{A} \times \vec{B}]$……. (15) because unit vector $\overrightarrow{a_N}$ reverses its direction.

2. The cross product is not associative.

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C} \ \ldots\ldots. (16)$$

3. With respect to addition cross product is distributive. Thus

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \ \ldots\ldots. (17)$$

4. If the two vectors are parallel to each other i.e. $\theta = 0^0$ then $\sin \theta_{AB} = 0$ thus

$$\vec{A} \times \vec{B} = 0 \ \ldots\ldots (18)$$

5. If the cross product of a vector with itself is performed, the result is zero.

$$\vec{A} \times \vec{A} = |\vec{A}||\vec{A}|\sin 0^0 = 0 \ \ldots\ldots.. (19)$$

6. Cross product of unit vectors:

$$\overrightarrow{a_x} \times \overrightarrow{a_y} = \overrightarrow{a_z} \ \ldots\ldots (20)$$

$$\overrightarrow{a_y} \times \overrightarrow{a_z} = \overrightarrow{a_x} \ \ldots\ldots (21)$$

$$\overrightarrow{a_z} \times \overrightarrow{a_x} = \overrightarrow{a_y} \ \ldots\ldots (22)$$

But if the order of unit vectors is reversed, the result is negative of the remaining third unit vector. Thus,

$$\overrightarrow{a_y} \times \overrightarrow{a_x} = -\overrightarrow{a_z}, \overrightarrow{a_z} \times \overrightarrow{a_y} = -\overrightarrow{a_x}, \overrightarrow{a_x} \times \overrightarrow{a_z} = -\overrightarrow{a_y} \ \ldots\ldots (23)$$

7. Any unit vector cross product with itself is zero

$$\overrightarrow{a_x} \times \overrightarrow{a_x} = \overrightarrow{a_y} \times \overrightarrow{a_y} = \overrightarrow{a_z} \times \overrightarrow{a_z} = 0 \ldots\ldots (24)$$

8. Cross product is determinant form.

If $\vec{A}$ and $\vec{B}$ are in Cartesian co-ordinate system.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \overrightarrow{a_x} & \overrightarrow{a_y} & \overrightarrow{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \ldots\ldots\ldots (25)$$

If $\vec{A}$ and $\vec{B}$ are in Cylindrical co-ordinate system.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \overrightarrow{a_\rho} & \overrightarrow{a_\phi} & \overrightarrow{a_z} \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix} \ldots\ldots\ldots (26)$$

If $\vec{A}$ and $\vec{B}$ are in Spherical co-ordinate system.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \overrightarrow{a_r} & \overrightarrow{a_\theta} & \overrightarrow{a_\phi} \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix} \ldots\ldots\ldots (27)$$

Example 1.11 : *Given the two coplanar vectors*

$$\bar{A} = 3\bar{a}_x + 4\bar{a}_y - 5\bar{a}_z \text{ and } \bar{B} = -6\bar{a}_x + 2\bar{a}_y + 4\bar{a}_z$$

Obtain the unit vector normal to the plane containing the vectors $\bar{A}$ *and* $\bar{B}$.

Solution : Note that the unit vector normal to the plane containing the vectors $\bar{A}$ and $\bar{B}$ is the unit vector in the direction of cross product of $\bar{A}$ and $\bar{B}$.

Now
$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 3 & 4 & -5 \\ -6 & 2 & 4 \end{vmatrix}$$

$$= \bar{a}_x \begin{vmatrix} 4 & -5 \\ 2 & 4 \end{vmatrix} - \bar{a}_y \begin{vmatrix} 3 & -5 \\ -6 & 4 \end{vmatrix} + \bar{a}_z \begin{vmatrix} 3 & 4 \\ -6 & 2 \end{vmatrix}$$

$$= 26\bar{a}_x + 18\bar{a}_y + 30\bar{a}_z$$

$$\therefore \quad \bar{a}_N = \frac{\bar{A} \times \bar{B}}{|\bar{A} \times \bar{B}|} = \frac{26\bar{a}_x - 18\bar{a}_y + 30\bar{a}_z}{\sqrt{(26)^2 + (18)^2 + (30)^2}}$$

$$= \mathbf{0.5964\ \bar{a}_x + 0.4129\ \bar{a}_y + 0.6882\ \bar{a}_z}$$

This is the unit vector normal to the plane containing $\bar{A}$ and $\bar{B}$.

1.7.6 Product of three vectors

Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ are the three given vectors. Then the product of these three vectors is classified in two ways called,

1. Scalar triple product

   $\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$....... (28)
2. Vector triple product

   $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$....... (29)

Example 1.12 : *The three fields are given by,*

$$\bar{A} = 2\bar{a}_x - \bar{a}_z \ , \quad \bar{B} = 2\bar{a}_x - \bar{a}_y + 2\bar{a}_z \ , \quad \bar{C} = 2\bar{a}_x - 3\bar{a}_y + \bar{a}_z$$

Find the scalar and vector triple product.

Solution : The scalar triple product is,

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix} = \mathbf{14}$$

The vector triple product is,

$$\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$$

$$\bar{A} \cdot \bar{C} = (2)(2) + (0)(-3) + (-1)(1) = 3$$

$$\bar{A} \cdot \bar{B} = (2)(2) + (0)(-1) + (-1)(2) = 2$$

$$\therefore \quad \bar{A} \times (\bar{B} \times \bar{C}) = 3\bar{B} - 2\bar{C} = 3[2\bar{a}_x - \bar{a}_y + 2\bar{a}_z] - 2[2\bar{a}_x - 3\bar{a}_y + \bar{a}_z]$$

$$= 2\ \bar{a}_x + 3\ \bar{a}_y + 4\ \bar{a}_z$$

1.8 Transformation of Vectors

Getting familiar with the dot and cross product, it is possible now to transform the vectors from one coordinate system to other co-ordinate system.

1.8.1 Transformation of vectors from Cartesian to Cylindrical and Cylindrical to Cartesian

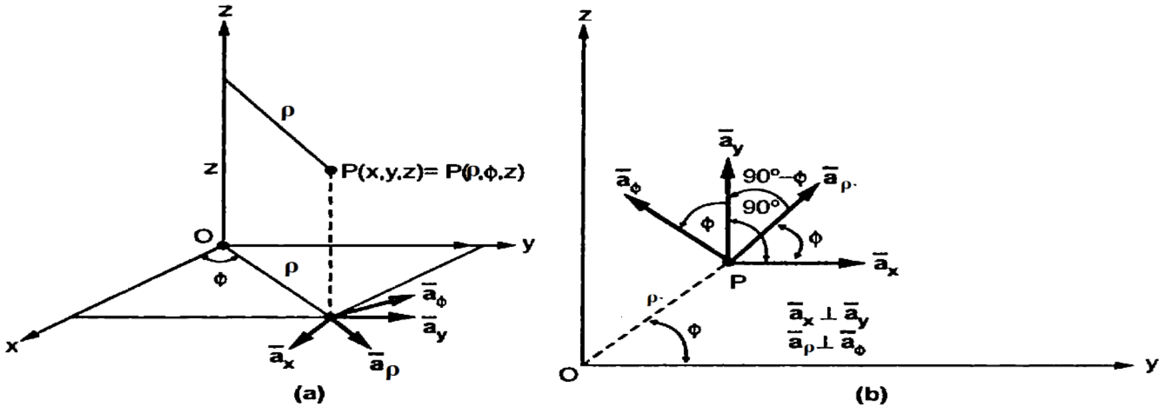


Fig. 1.8.1Transformation of vectors

Consider a vector $\vec{A}$ in cartesian co-ordinate system as,

$$\vec{A} = A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z} \ \ldots\ldots\ (1)$$

While the same vector in cylindrical co-ordinate system as

$$\vec{A} = A_\rho\overrightarrow{a_\rho} + A_\phi\overrightarrow{a_\phi} + A_z\overrightarrow{a_z} \ \ldots\ldots\ (2)$$

The result of transformation from cartesian to cylindrical can be expressed in matrix form as

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \ldots\ldots\ (3)$$

Similarly, the result of transformation from cylindrical to cartesian can be expressed in matrix form as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \ldots\ldots\ (4)$$

1.8.2 Transformation of vectors from Cartesian to Spherical and Spherical to Cartesian

Consider a vector $\vec{A}$ in cartesian co-ordinate system as,

$$\vec{A} = A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z} \ \ldots\ldots\ (5)$$

While the same vector in Spherical co-ordinate system as

$$\vec{A} = A_r\overrightarrow{a_r} + A_\theta\overrightarrow{a_\theta} + A_\phi\overrightarrow{a_\phi} \ \ldots\ldots\ (6)$$

The result of transformation from cartesian to cylindrical can be expressed in matrix form as

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \ldots\ldots (7)$$

Similarly, the result of transformation from cylindrical to cartesian can be expressed in matrix form as

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} \ldots\ldots (8)$$

1.8.3 Distances in all co-ordinate system

Consider two points A and B with the position vectors as, $\vec{A} = x_1\overrightarrow{a_x} + y_1\overrightarrow{a_y} + z_1\overrightarrow{a_z}$ and $\vec{B} = x_2\overrightarrow{a_x} + y_2\overrightarrow{a_y} + z_2\overrightarrow{a_z}$ then the distance d between the two points in all the three co-ordinate systems are given by,

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ ……Cartesian

$d = \sqrt{(r_2)^2 + (r_1)^2 - 2r_1r_2\cos(\emptyset_2 - \emptyset_1) + (z_2 - z_1)^2}$ ……Cylindrical

$d = \sqrt{(r_2)^2 + (r_1)^2 - 2r_1r_2\cos\theta_2\cos\theta_1 - 2r_1r_2\sin\theta_2\sin\theta_1 \, cos(\emptyset_2 - \emptyset_1)}$ ……Spherical

Example 1.17 : *Express vector $\overline{B}$ in cartesian and cylindrical systems.*
Given, $\overline{B} = \frac{10}{r}\overline{a}_r + r\cos\theta\overline{a}_\theta + \overline{a}_\phi$
Then find $\overline{B}$ at (– 3, 4, 0) and (5, π/2, – 2)

Solution :

$$\overline{B} = \frac{10}{r}\overline{a}_r + r\cos\theta\overline{a}_\theta + \overline{a}_\phi$$

$\therefore \quad B_r = \frac{10}{r}, \quad B_\theta = r\cos\theta, \quad B_\phi = 1$ ... in spherical

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \frac{10}{r} \\ r\cos\theta \\ 1 \end{bmatrix}$$

$\therefore \quad B_x = \frac{10}{r}\sin\theta\cos\phi + r\cos^2\theta\cos\phi - \sin\phi$ ... (1)

$\therefore \quad B_y = \frac{10}{r}\sin\theta\sin\phi + r\cos^2\theta\sin\phi + \cos\phi$ ... (2)

$\therefore \quad B_z = \frac{10}{r}\cos\theta - r\sin\theta\cos\theta$ ... (3)

But $\quad r = \sqrt{x^2 + y^2 + z^2}, \quad \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad \tan\phi = \frac{y}{x}$

$$\therefore \qquad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

Using equations (1), (2) and (3), $\overline{B}$ in cartesian system is :

$$\overline{B} = B_x \bar{a}_x + B_y \bar{a}_y + B_z \bar{a}_z \quad \text{where,}$$

$$B_x = \frac{10x}{x^2+y^2+z^2} + \frac{xz^2}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{y}{\sqrt{x^2+y^2}} \qquad \text{... (4)}$$

$$B_y = \frac{10y}{x^2+y^2+z^2} + \frac{yz^2}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + \frac{x}{\sqrt{x^2+y^2}} \qquad \text{... (5)}$$

$$B_z = \frac{10z}{x^2+y^2+z^2} - \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \qquad \text{... (6)}$$

At $(-3, 4, 0)$, $x = -3$, $y = 4$, $z = 0$

$$\therefore \qquad \overline{B} = -2\bar{a}_x + \bar{a}_y \qquad \text{... In cartesian}$$

For transforming spherical to cylindrical use,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\therefore \qquad B_\rho = \sin\theta\, B_r + \cos\theta\, B_\theta = \frac{10\sin\theta}{r} + r\cos^2\theta$$

$$B_\phi = B_\phi = 1$$

$$B_z = \cos\theta\, B_r - \sin\theta\, B_\theta = \frac{10\cos\theta}{r} - r\sin\theta\cos\theta$$

Now $\qquad \rho = r\sin\theta,\ z = r\cos\theta,\ \phi = \phi,\ r = \sqrt{\rho^2+z^2},\ \theta = \tan^{-1}\frac{\rho}{z}$

And $\qquad \tan\theta = \frac{\rho}{z}$ hence $\sin\theta = \frac{\rho}{\sqrt{\rho^2+z^2}}$, $\cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}$

$$\therefore \qquad \overline{B} = B_\rho \bar{a}_\rho + B_\phi \bar{a}_\phi + B_z \bar{a}_z \quad \text{where,}$$

$$B_\rho = \frac{10\rho}{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}}, \quad B_\phi = 1, \quad B_z = \frac{10z}{\rho^2+z^2} - \frac{\rho z}{\sqrt{\rho^2+z^2}}$$

At given point $\left(5, \frac{\pi}{2}, -2\right)$, $\rho = 5$, $\phi = \frac{\pi}{2}$ and $z = -2$

$$\therefore \qquad B_\rho = \frac{10\times 5}{5^2+(-2)^2} + \frac{(-2)^2}{\sqrt{5^2+(-2)^2}} = 2.467, \quad B_\phi = 1$$

$$B_z = \frac{10\times(-2)}{5^2+(-2)^2} - \frac{5\times(-2)}{\sqrt{5^2+(-2)^2}} = 1.167$$

$$\therefore \qquad \overline{B} = 2.467\,\bar{a}_\rho + \bar{a}_\phi + 1.167\,\bar{a}_z \qquad \text{... In cylindrical}$$

Chapter-2

Vector Calculus

2.1 Introduction

Vector calculus concern differential and integral operations involving vector functions. Electromagnetic fields are functions of position in space as well as time. So in Electromagnetic fields, we will be often required to perform line, surface and volume integrations. The evaluation of these integral requires knowledge of length, surface, volume.

2.2 Differential Elements in Cartesian Co-ordinate System

Consider a point P (x, y, z) in rectangular co-ordinate system. Let us increase each co-ordinate by a differential amount. A new point P' will be obtained having co-ordinates (x+dx, y+dy, z+dz).

Thus, dx = Differential length in x-direction
dy = Differential length in y-direction
dz = Differential length in z-direction

Hence differential vector length also called **elementary vector length** can be represented as,

$$\overrightarrow{dl} = dx\overrightarrow{a_x} + dy\overrightarrow{a_y} + dz\overrightarrow{a_z} \ldots\ldots (1)$$

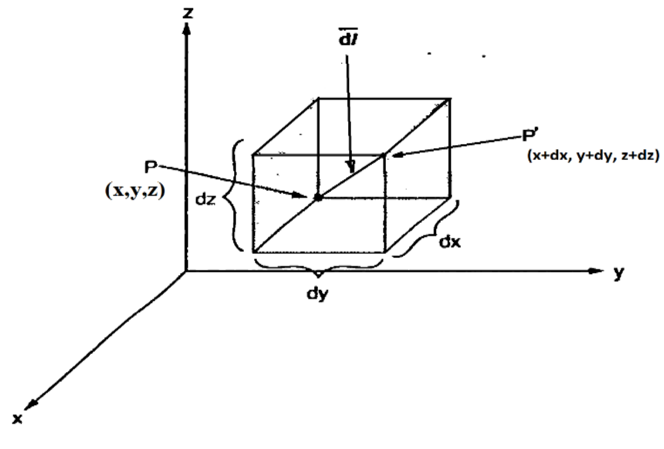


Fig. 2.2.1 Differential elements and different length in cartesian system

The distance P' from P is given by magnitude of the differential vector length,

$$\left|\overrightarrow{dl}\right| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \ldots\ldots (2)$$

Hence the **differential volume** of the rectangular parallelepiped is given by,

$$dv = dxdydz \ldots\ldots (3)$$

Note that $\overrightarrow{dl}$ is a vector but dv is a scalar.

The **differential surface area** element $\overrightarrow{dS}$ is represented as,

$$\overrightarrow{dS} = dS\overrightarrow{a_n} \ldots\ldots (4)$$

Where dS = the differential surface area of the element

$\overrightarrow{a_n}$ = unit vector normal to surface dS

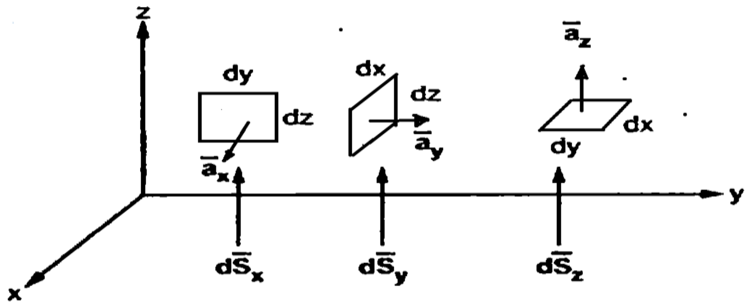


Fig.2.2.2 Differential surface elements in cartesian system

Thus various differential surface elements in cartesian co-ordinate system are shown in Fig. 2.2.2.

The vector representation of these elements are given as,

$\overrightarrow{dS_x}$ = Differential vector surface area normal to x-direction =dydz$\overrightarrow{a_x}$..... (5)

$\overrightarrow{dS_y}$ = Differential vector surface area normal to y-direction =dxdz$\overrightarrow{a_y}$..... (6)

$\overrightarrow{dS_z}$ = Differential vector surface area normal to z-direction =dxdy$\overrightarrow{a_z}$..... (7)

2.3 Differential Elements in Cylindrical Co-ordinate System

Consider a point P (ρ, ϕ, z) in cylindrical co-ordinate system. Let us increase each co-ordinate by a differential amount. A new point P' will be obtained having co-ordinates (ρ+dρ, ϕ+dϕ, z+dz).

Thus, dρ = Differential length in ρ-direction

ρdϕ = Differential length in ϕ-direction

dz = Differential length in z-direction

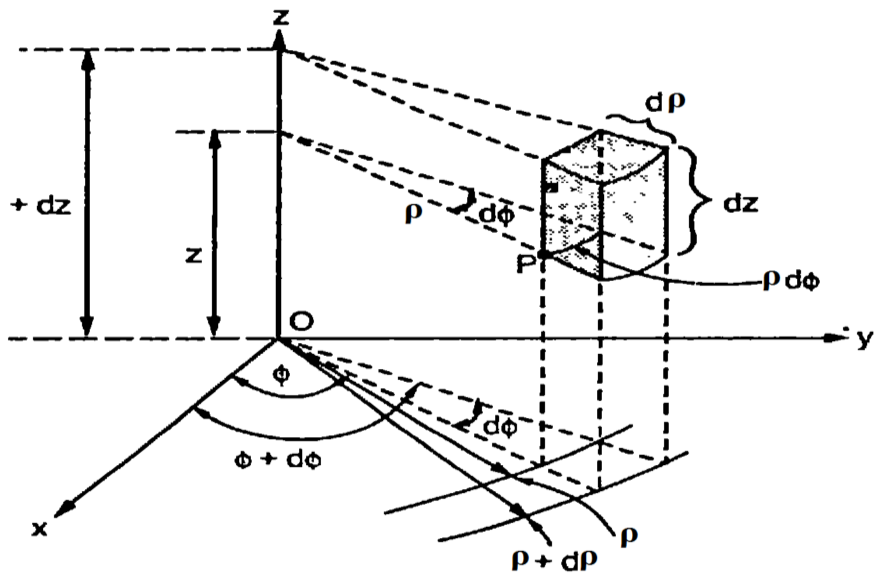


Fig. 2.3.1 Differential volume in cylindrical co-ordinate system

Hence differential vector length also called **elementary vector length** can be represented as,

$$\overrightarrow{dl} = d\rho\overrightarrow{a_\rho} + \rho d\phi\overrightarrow{a_\phi} + dz\overrightarrow{a_z} \ldots\ldots. (1)$$

The magnitude of the differential vector length,

$$\left|\overrightarrow{dl}\right| = \sqrt{(d\rho)^2 + (\rho d\phi)^2 + (dz)^2} \ldots\ldots. (2)$$

Hence the **differential volume** of the rectangular parallelepiped is given by,

$$dv = \rho d\rho d\phi dz \ldots\ldots. (3)$$

Note that $\overrightarrow{dl}$ is a vector but dv is a scalar.

The **differential surface areas** in three directions are shown in Fig. 2.3.2

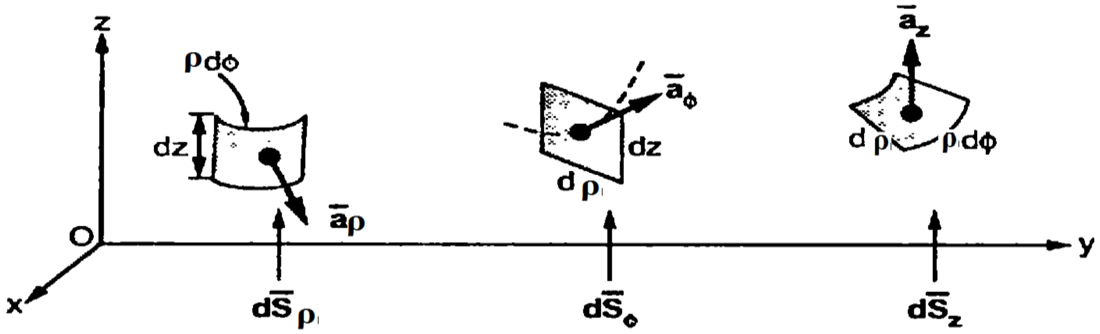


Fig. 2.3.2 Differential surface elements in cylindrical system

.

The vector representation of these differential elements are given as,

$\overrightarrow{dS_\rho}$ = Differential vector surface area normal to ρ-direction =ρdϕdz$\overrightarrow{a_\rho}$….. (4)

$\overrightarrow{dS_\phi}$ = Differential vector surface area normal to ϕ-direction =dρdz$\overrightarrow{a_\phi}$….. (5)

$\overrightarrow{dS_z}$ = Differential vector surface area normal to z-direction =ρdρdϕ$\overrightarrow{a_z}$….. (6)

2.4 Differential Elements in Spherical Co-ordinate System

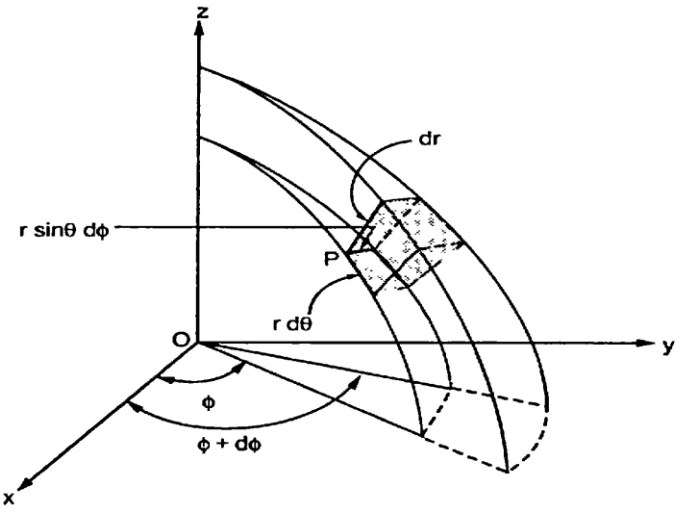


Fig. 2.4.1 Differential volume in spherical co-ordinate system

Consider a point P (r, θ, ϕ) in spherical co-ordinate system. Let us increase each co-ordinate by a differential amount. A new point P' will be obtained having co-ordinates (r+dr, θ+dθ, ϕ+dϕ).

Thus, dr = Differential length in r-direction
rdθ = Differential length in θ-direction
rsinθdϕ = Differential length in ϕ-direction

Hence differential vector length also called **elementary vector length** can be represented as,

$$\overrightarrow{dl} = dr\overrightarrow{a_r} + rd\theta\overrightarrow{a_\theta} + rsin\theta d\phi\overrightarrow{a_\phi} \ldots\ldots (1)$$

The magnitude of the differential vector length,

$$\left|\overrightarrow{dl}\right| = \sqrt{(dr)^2 + (rd\theta)^2 + (rsin\theta d\phi)^2} \ldots\ldots (2)$$

Hence the **differential volume** of the rectangular parallelepiped is given by,

$$dv = r^2 \sin\theta drd\theta d\emptyset \ldots\ldots (3)$$

The **differential surface areas** in three directions are shown in Fig. 2.4.2

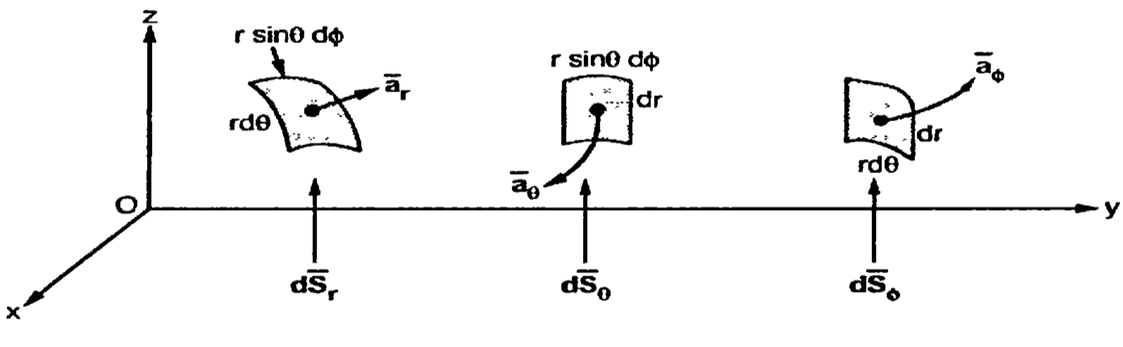


Fig. 2.4.2 Differential surface elements in spherical co-ordinate system

The vector representation of these differential elements are given as,

$\overrightarrow{dS_r}$ = Differential vector surface area normal to r-direction

$$= r^2 \sin\theta \, d\theta d\emptyset \overrightarrow{a_r} \ldots\ldots (4)$$

$\overrightarrow{dS_\theta}$ = Differential vector surface area normal to θ-direction

$$= r \sin\theta \, dr \, d\phi \overrightarrow{a_\theta} \ldots\ldots (5)$$

$\overrightarrow{dS_\phi}$ = Differential vector surface area normal to ϕ-direction $= r \, dr \, d\theta \overrightarrow{a_\phi} \ldots\ldots$ (6)

2.5 Types of Integral Related to Electromagnetic Theory

In electromagnetic theory a charge can exist in point form, line form, surface form or volume form. Hence while dealing with the analysis of such charge distributions, the various types of integrals are required. These types are,

1. Line Integral
2. Surface Integral
3. Volume Integral

2.5.1 Line Integral

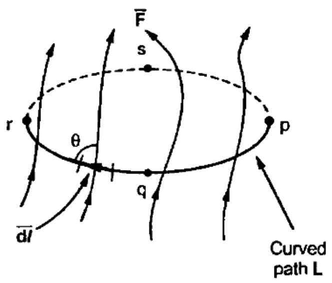


Fig. 2.5.1 Line integral

The line integral is the integral of the tangential component of a vector field $\vec{F}$ along the curved path L in space is given as

$$\int_L \vec{F}.\overrightarrow{dl} = \int_p^r |\vec{F}| \, dl \, \cos\theta \ldots. (1)$$

….. using definition of dot product

Where dl = Elementary length

This is called **line integral** of $\vec{F}$ around the closed path L. If the path of integration is a closed path or contour such as p-q-r-s-p, the corresponding integral is called **contour integral** or **closed integral** or **circular integral** and mathematically defined as,

$$\oint_L \vec{F}.\overrightarrow{dl} = \text{circular integral} \ldots\ldots (2)$$

The integral represents **circulation of the vector field $\vec{F}$ around the closed path L.**

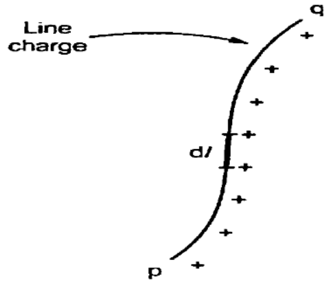


Fig. 2.5.2 Line charge

If there exist a charge along a line as shown in Fig. 2.5.2, then the total charge is obtained by calculating the line integral,

$$Q = \int_L \rho_L.\overrightarrow{dl} \ldots\ldots (3)$$

Where ρ_L = line charge density i.e. charge per unit length (C/m)

Key Point: In evaluating line integration, the $\overrightarrow{dl}$ direction is assumed to be always positive and limits of integration decide the sign of the integral.

2.5.2 Surface Integral

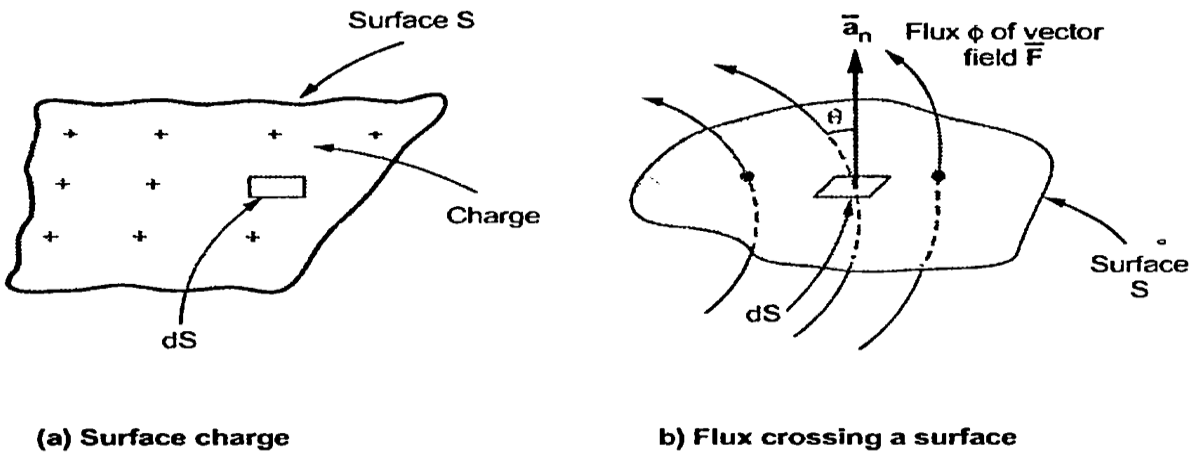


Fig. 2.5.3

Let's a vector field $\vec{F}$, continuous in a region containing the smooth surface S. We define the surface integral or the flux of $\vec{F}$ through S as ,

$$\Psi = \int_S \vec{F}.\overrightarrow{ds} = \int_S \left|\vec{F}\right| \cos\theta \, ds = \int_S \vec{F}.\overrightarrow{a_n}\, ds \;\ldots\ldots\; (4)$$

Where $\overrightarrow{a_n}$ = unit vector normal to the surface S

If the surface is closed, then it defines a volume and corresponding surface integral is given by,

$$\Psi = \oint_S \vec{F}.\overrightarrow{ds} \;\ldots\ldots\; (5)$$

This represents the net outward flux of vector field $\vec{F}$ from surface S.

Key Points:

1. The closed surface defines a volume.
2. The surface integral involves the double integration procedure mathematically.

For a charge distribution shown in Fig. 2.5.3 (a), we can write for the total charge existing on the surface as,

$$Q = \int_s \rho_S.\overrightarrow{ds} \;\ldots\ldots\; (6)$$

Where ρ_s = surface charge density in C/m^2

ds = elementary surface

2.5.3 Volume Integral

If the charge distribution exists in a three dimensional volume form as shown in Fig. 2.5.4 then a volume integral is required to calculate the total charge.

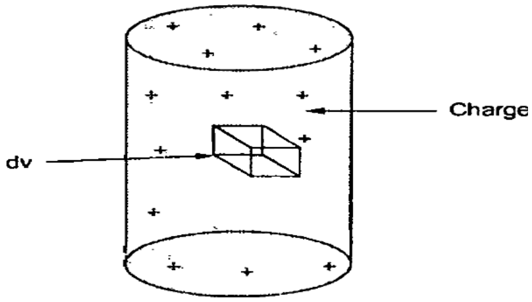


Fig. 2.5.4 Volume charge

Thus if ρ_v is the volume charge density over a volume v then the volume integral is defined as

$$Q = \int_v \rho_v dv \text{ …… (7)}$$

Where dv= elementary volume

ρ_v= volume charge density in C/m^3

Example 1.18 : *Calculate the circulation of vector field,*

$$\bar{F} = r^2 \cos\phi\, \bar{a}_r + z \sin\phi\, \bar{a}_z$$

around the path L defined by $0 \le r \le 3$, $0 \le \phi \le 45^\circ$ *and* $z = 0$ *as shown in the Fig. 1.48(a).*

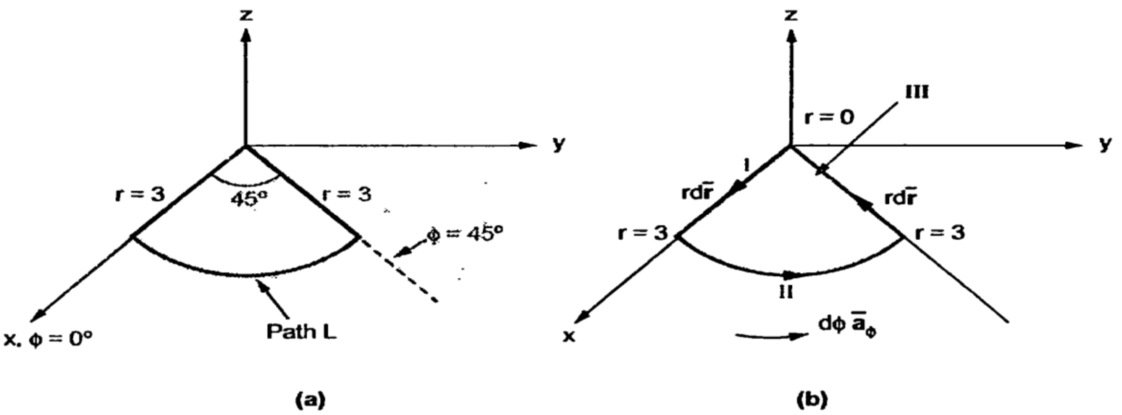


Fig. 1.48

Solution : Divide the given path L into three sections.

Section I : r varies from 0 to 3, $\phi = 0^\circ$ and $z = 0$

$\therefore \quad d\bar{l} = dr\, \bar{a}_r$ … Along radial direction

$$\therefore \int_{I} \bar{F}\cdot d\bar{l} = \int_{r=0}^{3} (r^2 \cos\phi\ \bar{a}_r + z \sin\phi\, \bar{a}_z)\cdot dr\, \bar{a}_r$$

$$= \int_{r=0}^{3} r^2 \cos\phi\ dr \quad \text{… } \bar{a}_r \cdot \bar{a}_r = 1,\ \bar{a}_z \cdot \bar{a}_r = 0$$

$$= \left[\frac{r^3}{3}\right]_0^3 \cos 0^\circ = \left[\frac{27}{3}\right][1] = 9$$

Section II : r is constant 3, ϕ varies from 0 to 45°, $z = 0$

$\therefore \quad d\bar{l} = d\phi\, \bar{a}_\phi$ … Along ϕ direction

$$\therefore \int_{II} \bar{F}\cdot d\bar{l} = \int_{\phi=0}^{45^\circ} (r^2 \cos\phi\ \bar{a}_r + z \sin\phi\, \bar{a}_z)\cdot d\phi\, \bar{a}_\phi$$

$$= 0 \quad \text{… } \bar{a}_r \cdot \bar{a}_\phi = \bar{a}_z \cdot \bar{a}_\phi = 0$$

Section III : r varies from 3 to 0, $\phi = 45^\circ$ and $z = 0$

$$d\bar{l} = dr\, \bar{a}_r$$

Note that $d\bar{l}$ is always positive, limits of integration from r = 3 to 0 taking care of direction.

$$\therefore \int_{III} \bar{F}\cdot d\bar{l} = \int_{r=3}^{0} (r^2 \cos\phi\ \bar{a}_r + z \sin\phi\, \bar{a}_z)\cdot dr\, \bar{a}_r$$

$$= \int_{r=3}^{0} r^2 \cos\phi \, dr \qquad \ldots \bar{a}_r \cdot \bar{a}_r = 1, \bar{a}_z \cdot \bar{a}_r = 0$$

$$= \cos 45^\circ \left[\frac{r^3}{3}\right]_3^0 = 0.7071 \left[\frac{-27}{3}\right] = -6.3639$$

$$\therefore \qquad \oint_L \bar{F} \cdot d\bar{l} = 9 + 0 - 6.3639 = 2.636$$

2.6 Del Operator

The del operator (∇) is a vector differential operator also known as **gradient operator** and is not a vector in itself. In Cartesian co-ordinates,

$$\nabla = \frac{\partial}{\partial x}\overrightarrow{a_x} + \frac{\partial}{\partial y}\overrightarrow{a_y} + \frac{\partial}{\partial z}\overrightarrow{a_z} \ldots\ldots (1)$$

The operator is useful in defining

1. The gradient of a scalar V, written as ∇V
2. The divergence of a vector $\vec{A}$, written as $\nabla . \vec{A}$
3. The curl of a vector $\vec{A}$, written as $\nabla \times \vec{A}$
4. The Laplacian of a scalar V, written as $\nabla^2 V$

In cylindrical co-ordinates as,

$$\nabla = \frac{\partial}{\partial \rho}\overrightarrow{a_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\overrightarrow{a_\phi} + \frac{\partial}{\partial z}\overrightarrow{a_z} \ldots\ldots (2)$$

And in spherical co-ordinates as,

$$\nabla = \frac{\partial}{\partial r}\overrightarrow{a_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\overrightarrow{a_\theta} + \frac{1}{r sin\theta}\frac{\partial}{\partial \phi}\overrightarrow{a_\phi} \ldots\ldots (3)$$

2.7 Gradient of a Scalar

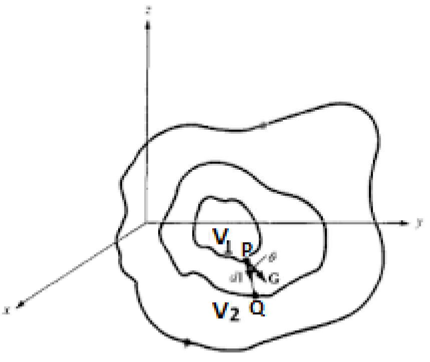


Fig. 2.7.1 Gradient of a scalar Field V

The gradient of a scalar field V is a vector that represents both the magnitude and direction of the maximum space rate of increase of V.

Mathematical expression for the gradient can be obtained by evaluating the difference in the field dV between points P and Q, where V_1, V_2 are contours on which V is constant.

Increase of scalar field V,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= \left(\frac{\partial V}{\partial x}\overrightarrow{a_x} + \frac{\partial V}{\partial y}\overrightarrow{a_y} + \frac{\partial V}{\partial z}\overrightarrow{a_z}\right).\left(dx\overrightarrow{a_x} + dy\overrightarrow{a_y} + dz\overrightarrow{a_z}\right)$$

Let, $\vec{G} = \left(\frac{\partial V}{\partial x}\overrightarrow{a_x} + \frac{\partial V}{\partial y}\overrightarrow{a_y} + \frac{\partial V}{\partial z}\overrightarrow{a_z}\right) \ldots\ldots (1)$

Then, $$dV = \vec{G}.\overrightarrow{dl} = Gdlcos\theta$$

$$\frac{dV}{dl} = G\cos\theta = \vec{G}.\overrightarrow{a_l} \text{ ...... (2)}$$

Where $\frac{dV}{dl}$ is the directional derivative along the direction $\overrightarrow{a_l}$

$\overrightarrow{a_l}$ is an unit vector along $\overrightarrow{dl}$ i.e. $\overrightarrow{a_l} = \frac{\overrightarrow{dl}}{dl}$

$\overrightarrow{dl}$ is the differential displacement from P to Q

θ is the angle between $\vec{G}$ and $\overrightarrow{dl}$

$\frac{dV}{dl}$ is maximum when θ = 0^0 i.e. $\overrightarrow{dl}$ is in the direction of $\vec{G}$ or distance *dl* from P to Q must be minimum or $\overrightarrow{a_l}$ is normal to the surface.

Hence $\left.\frac{dv}{dl}\right|_{max} = G\cos 0^0 = G$......... (3)

Thus $\vec{G}$ has its magnitude and direction as those of maximum rate of change of V. Therefore, for cartesian co-ordinate system,

$$grad\ V = \nabla V = \frac{\partial V}{\partial x}\overrightarrow{a_x} + \frac{\partial V}{\partial y}\overrightarrow{a_y} + \frac{\partial V}{\partial z}\overrightarrow{a_z} \text{....... (4)}$$

For cylindrical co-ordinate system,

$$grad\ V = \nabla V = \frac{\partial V}{\partial \rho}\overrightarrow{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\overrightarrow{a_\phi} + \frac{\partial V}{\partial z}\overrightarrow{a_z} \text{....... (5)}$$

For spherical co-ordinate system,

$$grad\ V = \nabla V = \frac{\partial V}{\partial r}\overrightarrow{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\overrightarrow{a_\theta} + \frac{1}{rsin\theta}\frac{\partial V}{\partial \phi}\overrightarrow{a_\phi} \text{....... (6)}$$

Key Point: Gradient of scalar is a vector.

2.7.1 Properties of Gradient of a Scalar

The various properties of a gradient of a scalar field V are,

1. The gradient ∇V gives the maximum rate of change of V per unit distance.
2. The gradient ∇V always indicates the direction of the maximum rate of change of V.
3. The gradient ∇V at any point is perpendicular to the constant V surface, which passes through the point.
4. The directional derivative of V along the unit vector $\overrightarrow{a_l}$ is $V.\overrightarrow{a_l}$ which is projection of ∇V in the direction of unit vector $\overrightarrow{a_l}$.
5. $\nabla(V + U) = \nabla V + \nabla U$ where U is another scalar field
6. $\nabla(VU) = V\nabla U + U\nabla V$
7. $\nabla\left(\frac{V}{U}\right) = \frac{U\nabla V - V\nabla U}{U^2}$
8. $\nabla V^n = nV^{n-1}\nabla V$

Example 1.20 : *A particular scalar field α is given by,*

a) $\alpha = 20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$ ... *In cart*

b) $\alpha = 25\, r \sin\phi$ ... *In cylin*

c) $\alpha = \frac{40\cos\theta}{r^2}$ ... *In sph*

Find its gradient at $P(0,1,1)$ for cartesian, $P\left(\sqrt{2}, \frac{\pi}{2}, 5\right)$ for cylindrical $P(3, 60°, 30°)$ for the spherical.

Solution : a) $\alpha = 20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)$ in cartesian

$$\nabla\alpha = \frac{\partial\alpha}{\partial x}\bar{a}_x + \frac{\partial\alpha}{\partial y}\bar{a}_y + \frac{\partial\alpha}{\partial z}\bar{a}_z$$

$$\frac{\partial\alpha}{\partial x} = \frac{\partial}{\partial x}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = -20\, e^{-x}\sin\left(\frac{\pi y}{6}\right)$$

$$\frac{\partial\alpha}{\partial y} = \frac{\partial}{\partial y}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = 20\, e^{-x}\cos\left(\frac{\pi y}{6}\right)\times\frac{\pi}{6}$$

$$\frac{\partial\alpha}{\partial z} = \frac{\partial}{\partial z}\left[20\, e^{-x} \sin\left(\frac{\pi y}{6}\right)\right] = 0$$

$$\therefore \quad \nabla\alpha = -20\, e^{-x}\sin\left(\frac{\pi y}{6}\right)\bar{a}_x + 20\, e^{-x}\frac{\pi}{6}\cos\left(\frac{\pi y}{6}\right)\bar{a}_y$$

$\therefore$ At P (0, 1, 1) the $\nabla\alpha = -10\,\bar{a}_x + 9.0689\,\bar{a}_y$

b) $\alpha = 25\, r \sin\phi$ in cylindrical.

$$\therefore \quad \nabla\alpha = \frac{\partial\alpha}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial\alpha}{\partial\phi}\bar{a}_\phi + \frac{\partial\alpha}{\partial z}\bar{a}_z$$

$$\frac{\partial\alpha}{\partial r} = 25\sin\phi \qquad \frac{\partial\alpha}{\partial\phi} = 25\, r\cos\phi, \qquad \frac{\partial\alpha}{\partial z} = 0$$

$$\therefore \quad \nabla\alpha = 25\sin\phi\,\bar{a}_r + 25\cos\phi\,\bar{a}_\phi$$

$\therefore$ At $P\left(\sqrt{2}, \frac{\pi}{2}, 5\right)$ the $\nabla\alpha = 25\,\bar{a}_r$

c) $\alpha = \frac{40\cos\theta}{r^2}$ in spherical.

$$\therefore \quad \nabla\alpha = \frac{\partial\alpha}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial\alpha}{\partial\theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial\alpha}{\partial\phi}\bar{a}_\phi$$

$$\frac{\partial\alpha}{\partial r} = 40\cos\theta\left[-2\, r^{-3}\right] = -80\,\frac{\cos\theta}{r^3}$$

$$\frac{\partial\alpha}{\partial\theta} = -\frac{40}{r^2}\sin\theta, \qquad \frac{\partial\alpha}{\partial\phi} = 0$$

$$\therefore \quad \nabla\alpha = -\frac{80\cos\theta}{r^3}\bar{a}_r - \frac{40}{r^3}\sin\theta\,\bar{a}_\theta$$

$\therefore$ **At P (3, 60°, 30°) the $\nabla\alpha = -1.4814\,\bar{a}_r - 0.9362\,\bar{a}_\theta$**

Example: Given $W = x^2y^2 + xyz$, compute ∇W and the directional derivative $\frac{dW}{dl}$ in the direction $3\overrightarrow{a_x} + 4\overrightarrow{a_y} + 12\overrightarrow{a_z}$ at (2, -1, 0).

Ans: $W = x^2y^2 + xyz$, $\nabla W = \frac{\partial W}{\partial x}\overrightarrow{a_x} + \frac{\partial W}{\partial y}\overrightarrow{a_y} + \frac{\partial W}{\partial z}$

$$=(2xy^2 + yz)\overrightarrow{a_x} + (2x^2y + xz)\overrightarrow{a_y} + xy\overrightarrow{a_z}$$

At (2, -1, 0), $\nabla W = 4\overrightarrow{a_x} - 8\overrightarrow{a_y} - 2\overrightarrow{a_z}$

$$\frac{dW}{dl} = \nabla W.\overrightarrow{a_l} = \left(4\overrightarrow{a_x} - 8\overrightarrow{a_y} - 2\overrightarrow{a_z}\right).\frac{3\overrightarrow{a_x} + 4\overrightarrow{a_y} + 12\overrightarrow{a_z}}{\sqrt{3^2 + 4^2 + 12^2}} = \frac{12 - 32 - 24}{13}$$

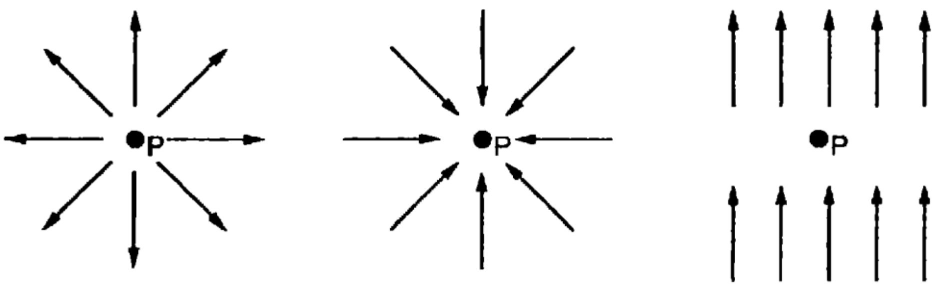
$$= -\frac{44}{13}$$

2.8 Divergence of a Vector

As we know, the net outflow of the flux of a vector field $\vec{A}$ from a closed surface S is obtained as $\oint \vec{A}.\overrightarrow{dS}$. The divergence of $\vec{A}$ as the net outward flow of flux per unit volume over a closed incremental surface. It is denoted as div $\vec{A}$ and given by,

$$div\, \vec{A} = \nabla.\vec{A} = \lim_{\Delta v \to 0} \frac{\oint \vec{A}.\overrightarrow{dS}}{\Delta v} \text{......} (1)$$

Where Δv is the differential volume element



(a) Positive e.g Electric Field at positive Chrage **(b) Negative** e.g Electric Field at negative Chrage **(c) Zero**

Fig. 2.8.1 Divergence at P

Key Point: Divergence of a vector field $\vec{A}$ at a point, P is the outward flux per unit volume as the volume shrinks about point P i.e. lim $\Delta v \to 0$ representing differential volume element at point P.

Symbolically it is denoted as,

$$\nabla.\vec{A} = \text{Divergence of } \vec{A}$$

Where ∇=vector operator $= \frac{\partial}{\partial x}\overrightarrow{a_x} + \frac{\partial}{\partial y}\overrightarrow{a_y} + \frac{\partial}{\partial z}\overrightarrow{a_z}$

But $\vec{A} = A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z}$

Hence $\nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ =div of $\vec{A}$ ….. (2)

This is the divergence of $\vec{A}$ in cartesian system.
For a cylindrical coordinate system,

$$\nabla.\vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} = \text{div of } \vec{A} \text{ ..... (3)}$$

For the spherical coordinate system,

$$\nabla.\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} = \text{div of } \vec{A} \text{ ..... (4)}$$

Physically divergence at a point indicates how much that vector field diverges from that point.

Consider a solenoid i.e. electromagnet obtained by winding a coil around the core. When current passes through it, flux is produced around it. Such a flux completes a closed path through the solenoid hence solenoidal field does not diverge. Thus mathematically, **the vector field having its divergence zero is called the solenoidal field.** For example, solenoidal fields are velocity field of an incompressible fluids, magnetic fields, conduction current density under steady state condition.

Hence $\nabla.\vec{A}=0$ for $\vec{A}$ to be solenoidal …….. (5)

2.8.1 Divergence Theorem

It is known that,

$$\nabla.\vec{A} = \lim_{\Delta v\to 0}\frac{\oint_S \vec{A}.\overrightarrow{dS}}{\Delta v} \text{ ……… Definition of divergence}$$

From the definition of divergence theorem, it can be written that,

$$\oint_S \vec{A}.\overrightarrow{dS} = \int_v (\nabla.\vec{A})dv \text{ …….. (6)}$$

This equation (6) is known as **divergence theorem.**

The Divergence theorem states that,

The Integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface.

The theorem can be applied to any vector field but partial derivatives of that vector field must exist. The divergence theorem as applied to the flux density, both sides of the divergence theorem give the net charge enclosed by the closed surface i.e. net flux crossing the closed surface.

Key Point: The divergence theorem converts the surface integral into a volume integral, provided that the closed surface encloses a certain volume. This is advantageous in electromagnetic theory as volume integrals are more easy to evaluate than surface integrals.

Example 1.19 : *A particular vector field $\bar{F} = r^2 \cos^2 \phi\, \bar{a}_r + z \sin \phi\, \bar{a}_\phi$ is in cylindrical system. Find the flux emanating due to this field from the closed surface of the cylinder $0 \le z \le 1$, $r = 4$. Verify the divergence theorem.*

Solution : The outward flux is given by,

$$\phi = \oint_S \bar{F} \cdot d\bar{S} \text{ over a closed surface S}$$

The cylindrical surface is shown in the Fig. 1.50.

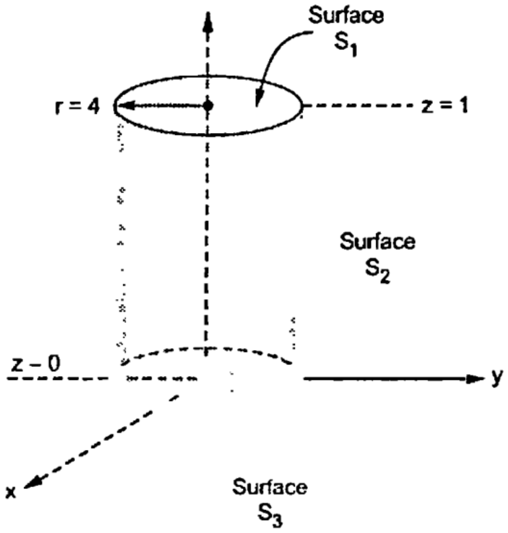


Fig. 1.50

The total surface is made up of,

1. Top surface S_1 for which z = 1, r varies from 0 to 4 and ϕ varies from 0 to 2π.
2. Lateral surface for which z varies from 0 to 1, ϕ from 0 to 2π and r = 4.
3. Bottom surface S_3 for which z = 0, r varies from 0 to 4 and ϕ varies from 0 to 2π.

For S_1, $\quad d\bar{S} = r\, dr\, d\phi\, \bar{a}_z$

For S_2, $\quad d\bar{S} = r\, dz\, d\phi\, \bar{a}_r$

For S_3, $\quad d\bar{S} = r\, dr\, d\phi\, (-\bar{a}_z)$

$$\therefore \quad \oint_{S_1} \bar{F} \cdot d\bar{S} = \oint_{S_1} (r^2 \cos^2 \phi\, \bar{a}_r + z \sin \phi\, \bar{a}_\phi) \cdot (r\, dr\, d\phi\, \bar{a}_z)$$

$$= 0 \qquad \ldots \bar{a}_r \cdot \bar{a}_z = \bar{a}_\phi \cdot \bar{a}_z = 0$$

$$\oint_{S_3} \bar{F} \cdot d\bar{S} = \oint_{S_3} (r^2 \cos^2 \phi\, \bar{a}_r + z \sin \phi\, \bar{a}_\phi) \cdot [r\, dr\, d\phi\, (-\bar{a}_z)]$$

$$= 0 \qquad \ldots \bar{a}_r \cdot \bar{a}_z = \bar{a}_\phi \cdot \bar{a}_z = 0$$

$$\oint_{S_2} \bar{F} \cdot d\bar{S} = \oint_{S_2} (r^2 \cos^2 \phi\, \bar{a}_r + z \sin \phi\, \bar{a}_\phi) \cdot (r\, dz\, d\phi\, \bar{a}_r)$$

$$= \int_{z=0}^{1}\int_{\phi=0}^{2\pi} r^2 \cos^2 \phi r\, dz\, d\phi \qquad \ldots\ \bar{a}_r \bullet \bar{a}_z = 1,\ \bar{a}_\phi \bullet \bar{a}_r = 0\ r = 4$$

$$= (4)^3 \int_{z=0}^{1}\int_{\phi=0}^{2\pi} dz \cos^2 \phi\ d\phi = 64 \int_0^1 dz \int_{\phi=0}^{2\pi} \frac{1+\cos 2\phi}{2} d\phi$$

$$= 64 \times [z]_0^1 \times \frac{1}{2} \times \left\{ [\phi]_0^{2\pi} + \left[+ \frac{\sin 2\phi}{2} \right]_0^{2\pi} \right\}$$

$$= 64 \times 1 \times \frac{1}{2} \times 2\pi = 64\pi$$

$$\therefore \qquad \oint_S \bar{F} \bullet d\bar{S} = 0 + 64\pi + 0 = 64\pi$$

Let us verify divergence theorem which states that,

$$\oint_S \bar{F} \bullet d\bar{S} = \oint_v (\nabla \bullet \bar{F})\, dv$$

Now $\qquad dv = r\, dr\, d\phi\, dz$

$$\nabla \bullet \bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(r\, F_r) + \frac{1}{r}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r \times r^2 \cos^2 \phi) + \frac{1}{r}\frac{\partial}{\partial \phi}(z \sin \phi) + 0$$

$$= \frac{\cos^2 \phi}{r} \times 3r^2 + \frac{z}{r}(+\cos \phi) = 3\, r \cos^2 \phi + \frac{z \cos \phi}{r}$$

$$\therefore \qquad \oint_v (\nabla \bullet \bar{F})\, dv = \int_{z=0}^{1}\int_{\phi=0}^{2\pi}\int_{r=0}^{4} \left(3r \cos^2 \phi + \frac{z \cos \phi}{r} \right) r\, dr\, d\phi\, dz$$

$$= \int_{z=0}^{1}\int_{\phi=0}^{2\pi} \left[\frac{3r^3}{3} \cos^2 \phi + z \cos \phi\, r \right]_0^4 d\phi\, dz$$

$$= \int_{z=0}^{1}\int_{\phi=0}^{2\pi} \left\{ 4^3 \left[\frac{1+\cos 2\phi}{2} \right] + 4z \cos \phi \right\} d\phi\, dz$$

$$= \int_{z=0}^{1} \left\{ 32 \left[\phi + \frac{\sin 2\phi}{2} \right]_0^{2\pi} + 4z [\sin \phi]_0^{2\pi} \right\} dz$$

$$= \int_{z=0}^{1} \{32 \times [2\pi + 0] + 4z[0]\}\, dz = \int_{z=0}^{1} 64\pi\, dz$$

$$= 64\, \pi [z]_0^1 = 64\pi$$

Thus $\int_S \bar{F} \bullet d\bar{S} = \oint_v (\nabla \bullet \bar{F})\, dv$ and divergence theorem is verified.

2.8.2 Properties of Divergence of a Vector

1. It produces a scalar field.
2. The divergence of a scalar V makes no scence.
3. $\nabla.\left(\vec{A} + \vec{B}\right) = \nabla.\vec{A} + \nabla.\vec{B}$
4. $\nabla.\left(V\vec{A}\right) = V\left(\nabla.\vec{A}\right) + \vec{A}.(\nabla V)$

2.9 Curl of a Vector

The curl of a vector field $\vec{A}$ is an axial (or rotational) vector at point P whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

$$Curl\,\vec{A} = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \left(\frac{\oint_L \vec{A}.\vec{dl}}{\Delta S} \right)_{max} \overrightarrow{a_n} \text{ ...... (1)}$$

Where ΔS = area enclosed by the line integral in normal direction

$\overrightarrow{a_n}$ = the unit vector normal to the surface ΔS and is determined using right hand rule.

Key Point: Curl indicates the rotational property of vector field. **If curl of vector is zero, the vector field is irrotational.** For example, electrostatic field and gravitational field.

In various coordinate systems, the curl of $\vec{A}$ is given by,

$$\nabla \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\overrightarrow{a_x} + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\overrightarrow{a_y} + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\overrightarrow{a_z}$$

$$\nabla \times \vec{A} = \begin{vmatrix} \overrightarrow{a_x} & \overrightarrow{a_y} & \overrightarrow{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \text{......Cartesian.... (2)}$$

$$\nabla \times \vec{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\overrightarrow{a_\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\overrightarrow{a_\phi} + \left[\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right]\overrightarrow{a_z}$$

$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \overrightarrow{a_\rho} & \rho\overrightarrow{a_\phi} & \overrightarrow{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} \text{......Cylindrical.... (3)}$$

$$\nabla \times \vec{A} = \frac{1}{rsin\theta}\left[\frac{\partial A_\phi sin\theta}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\overrightarrow{a_r} + \frac{1}{r}\left[\frac{1}{sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\overrightarrow{a_\theta} + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\overrightarrow{a_\phi}$$

$$\nabla \times \vec{A} = \frac{1}{r^2 sin\theta}\begin{vmatrix} \overrightarrow{a_r} & r\overrightarrow{a_\theta} & rsin\theta\overrightarrow{a_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & rsin\theta A_\phi \end{vmatrix} \text{......Spherical.... (4)}$$

2.9.1 Properties of Curl

1. Curl of a vector field is another vector field.
2. Curl of a scalr field makes no sense.
3. $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
4. $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla.\vec{B}) - \vec{B}(\nabla.\vec{A}) + (\vec{B}.\nabla)\vec{A} - (\vec{A}.\nabla)\vec{B}$
5. $\nabla \times (V\vec{A}) = V(\nabla \times \vec{A}) + \nabla V \times \vec{A}$
6. The divergence of curl of a vector field vanishes i.e. $\nabla.(\nabla \times \vec{A}) = 0$
7. The curl of the gradient of a scalar field vanishes i.e. $\nabla \times (\nabla V) = 0$

2.9.2 Stoke's Theorem

Stoke's theorem relates the line integral to surface integral. It states that, The line integral of $\vec{A}$ around a closed path, L is equal to the integral of the curl of $\vec{A}$ over the open surface S enclosed by the closed path L.

Mathematically it is expressed as,

$$\oint_L \vec{A}.\overrightarrow{dL} = \int_S (\nabla \times \vec{A}).\overrightarrow{dS}$$

Where dL= perimeter of total surface S

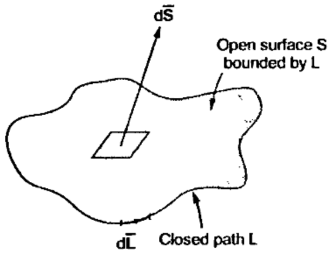


Fig.2.9.1

Key Point: Stoke's theorem is applicable only when $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on the surface S. The path L and open surface S enclosed by path L for which Stoke's theorem is applicable are shown in Fig. 2.9.1

2.9.3 Proof of Stoke's Theorem

Consider a surface S which is split into a number of incremental surfaces. Each incremental surface is having area ΔS as shown in Fig. 2.9.2.

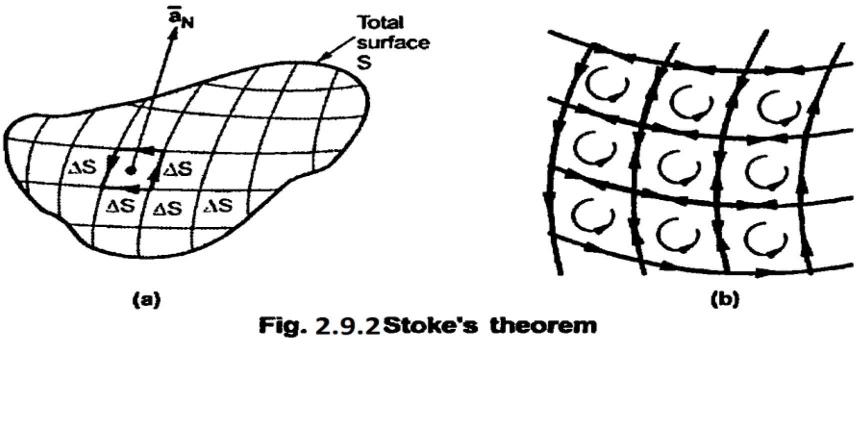


Fig. 2.9.2 Stoke's theorem

Applying the definition of curl to any of theses incremental surfaces we can write,

$$\left(\nabla \times \vec{H}\right)_n = \frac{\oint \vec{H}.\overrightarrow{dl}_{\Delta S}}{\Delta S} \ldots\ldots (1)$$

Where n=normal to ΔS according to the right-hand rule

$\overrightarrow{dl}_{\Delta S}$= perimeter of the incremental surface ΔS

Now the curl of $\vec{H}$ in the normal direction is the dot product of curl of $\vec{H}$ with $\overrightarrow{a_n}$ where $\overrightarrow{a_n}$ is the unit vector, normal to surface ΔS, according to the right-hand rule.

$$\left(\nabla \times \vec{H}\right)_n = \left(\nabla \times \vec{H}\right).\overrightarrow{a_n} \ldots\ldots \text{Using equation (1)}$$

$$\oint \vec{H}.\overrightarrow{dl}_{\Delta S} = \left(\nabla \times \vec{H}\right).\overrightarrow{a_n}\Delta S$$

$$\oint \vec{H}.\overrightarrow{dl}_{\Delta S} = \left(\nabla \times \vec{H}\right).\overrightarrow{\Delta S} \ldots\ldots. (2)$$

Hence obtain total curl for every incremental surface, add the closed line integrals for each ΔS. From the Fig.2.9.2, it can be seen that a common boundary between the two incremental surfaces, the line integral is getting canceled as the boundary is getting traced in two opposite directions.

This happens to all the interior boundaries. Only at the outside boundary cancelation does not exist. Hence summation of all closed line integrals for each and every ΔS ends up in a single closed line integral to be obtained for the outer boundary of the total surface S.

Hence the equation (2) becomes,

$$\oint_L \vec{H}.\overrightarrow{dL} = \int_S \left(\nabla \times \vec{H}\right).\overrightarrow{dS} \ldots\ldots (3)$$

Where dL = perimeter of the total surface S

Thus line integral can be expressed as a surface integral that proves Stoke's theorem.

Key Point: The Stoke's theorem is applicable for the open surface closed by the given closed path. Any Volume is a closed surface and hence application of Stoke's theorem to a closed surface that encloses certain volume produces zero answers.

➡ Example 1.21 : *Verify Stoke's theorem for a vector field*

$$\bar{F} = r^2 \cos\phi\ \bar{a}_r + z \sin\phi\ \bar{a}_z$$

around the path L defined by $0 \leq r \leq 3$, $0 \leq \phi \leq 45^\circ$ *and* $z = 0$ *as shown in the Fig. 1.52.*

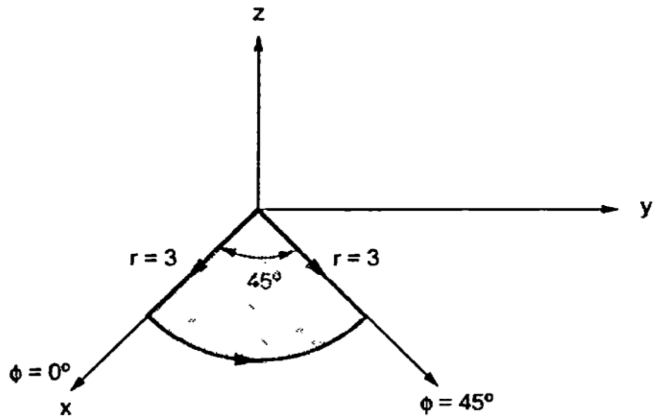


Fig. 1.52

Solution : From Stoke's theorem,

$$\oint_L \bar{F} \cdot d\bar{L} = \int_S (\nabla \times \bar{F}) \cdot d\bar{S}$$

The L.H.S. of Stoke's theorem is already evaluated in Ex. 1.18, which is **2.636**. (Refer Page 1-50).

To evaluate R.H.S., find $\nabla \times \bar{F}$

$$\nabla \times \bar{F} = \left[\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right]\bar{a}_r + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\bar{a}_\phi + \left[\frac{1}{r}\frac{\partial (r F_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right]\bar{a}_z$$

$$F_r = r^2 \cos\phi, \quad F_\phi = 0, \quad F_z = z \sin\phi$$

$$\therefore \quad \nabla \times \bar{F} = \left[\frac{1}{r} \times 0 - 0\right]\bar{a}_r + [0 - 0]\,\bar{a}_\phi + \left[\frac{1}{r}(0) - \frac{1}{r}(r)^2(-\sin\phi)\right]\bar{a}_z$$

$$= r \sin\phi\ \bar{a}_z$$

$d\bar{S} = r\, dr\, d\phi\ \bar{a}_z$ as surface is in x-y plane i.e. $z = 0$ plane for which normal direction is $\bar{a}_z$.

$$\therefore \int_S (\nabla \times \bar{F}) \cdot d\bar{S} = \int_S (r \sin\phi\, \bar{a}_z) \cdot (r\, dr\, d\phi)\, \bar{a}_z = \int_{\phi=0}^{45^\circ} \int_{r=0}^{3} r^2 \sin\phi\, dr\, d\phi = \left[\frac{r^3}{3}\right]_0^3 [-\cos\phi]_0^{45^\circ}$$

$$= [9]\,[-0.707 - (-1)] = 9 \times 0.2928 = \mathbf{2.636}$$

Thus Stoke's theorem is verified.

Example 7.13 : *Evaluate both sides of the Stoke's theorem for the field $\bar{H} = 6xy\bar{a}_x - 3y^2\bar{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\bar{S}$ be $\bar{a}_z$.*

Solution : According to Stoke's theorem,

$$\oint_L \bar{H} \cdot d\bar{L} = \int_S (\nabla \times \bar{H}) \cdot d\bar{S}$$

Let us evaluate left hand side. The integral to be evaluated on a perimeter of a closed path shown in the Fig. 7.39. The direction is a-b-c-d-a such that normal to it is positive $\bar{a}_z$ according to right hand rule.

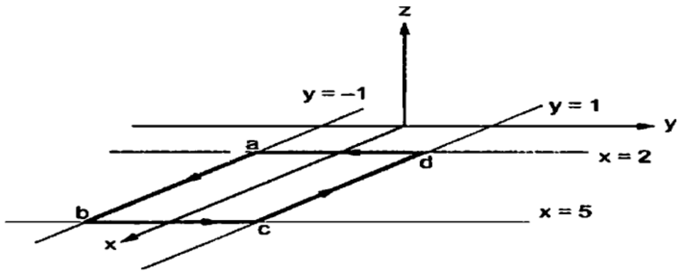


Fig. 7.39

$$\oint \bar{H} \cdot d\bar{L} = \int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \bar{H} \cdot d\bar{L}$$

$$\int_{ab} \bar{H} \cdot d\bar{L} = \int_{x=2}^{5} (6xy\,\bar{a}_x - 3y^2\,\bar{a}_y) \cdot dx\,\bar{a}_x$$

$$= \int_{x=2}^{5} 6xy\ dx = 6y \left[\frac{x^2}{2}\right]_2^5$$

$$= \frac{6y}{2}[25-4] = 63\,y$$

Now $y = -1$ for path ab, $\int_{ab} \bar{H} \cdot d\bar{L} = 63(-1) = -63$

Similarly $\int_{bc} \bar{H} \cdot d\bar{L} = \int_{y=-1}^{1} -3y^2\ dy = \frac{-3y^3}{3} = -[y^3]_{-1}^{1} = -[1-(-1)] = -2$

$$\int_{cd} \bar{H} \cdot d\bar{L} = \int_{x=5}^{2} 6xy\,dx = 6\left[\frac{x^2}{2}\right]_5^2 (y) = \frac{6y}{2}[4-25] = -63y$$

But $y = 1$ for path cd hence $\int_{cd} \bar{H} \cdot d\bar{L} = -63$

$$\int_{da} \bar{H} \cdot d\bar{L} = \int_{y=1}^{-1} -3y^2\ dy = -[y^3]_1^{-1} = -[(-1)^3 - (1)^3] = -[-1-1] = +2$$

$$\therefore \quad \oint \bar{H} \cdot d\bar{L} = -63-2-63+2 = -126\ \text{A}$$

Now evaluate right hand side.

$$\nabla \times \bar{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & -3y^2 & 0 \end{vmatrix}$$

$$= \bar{a}_x\,[0-0] + \bar{a}_y\,[0-0] + \bar{a}_z\,[0-6x]\,\bar{a}_z = -6x\bar{a}_z$$

$$\therefore \quad \int_S (\nabla \times \bar{H}) \cdot d\bar{S} = \int_S (-6x\,\bar{a}_z) \cdot (dx\,dy\,\bar{a}_z)$$

$$d\bar{S} = dx\ dy\ \bar{a}_z \quad \text{normal to direction } \bar{a}_z$$

$$\therefore \quad \int_S (\nabla \times \bar{H}) \cdot d\bar{S} = \int_{y=-1}^{1}\int_{x=2}^{5} -6x\,dx\,dy = -6\left[\frac{x^2}{2}\right]_2^5 [y]_{-1}^{1}$$

$$= -\frac{6}{2}[25-4][1-(-1)] = -3 \times 21 \times 2 = -126\ \text{A}$$

Thus both the sides are same, hence Stoke's theorem is verified.

Example 7.12 : *Prove that divergence of a curl of a vector is zero, using Stoke's theorem.*

Solution : Consider a vector $\bar{A}$.

The curl of $\bar{A}$ is $\nabla \times \bar{A}$ and its divergence is $\nabla \cdot (\nabla \times \bar{A})$. Now $\nabla \times \bar{A}$ is a vector while divergence of a vector is a scalar say α.

$$\therefore \quad \nabla \cdot (\nabla \times \bar{A}) = \alpha \qquad \ldots (1)$$

Let us evaluate integral of both sides over a volume

$$\therefore \quad \int_{vol} \nabla \cdot (\nabla \times \bar{A})\, dv = \int_{vol} \alpha \; dv \qquad \ldots (2)$$

Applying divergence theorem, the left hand side can be converted to a surface integral.

$$\therefore \quad \int_{vol} \nabla \cdot (\nabla \times \bar{A})\, dv = \int_{S} (\nabla \times \bar{A}) \cdot d\bar{S} \qquad \ldots (3)$$

where the S is closed surface enclosing the given volume.

$$\therefore \quad \int_{S} (\nabla \times \bar{A}) \cdot d\bar{S} = \int_{vol} \alpha \; dv \qquad \ldots (4)$$

Now if Stoke's theorem is applied, it can be seen that surface S on left hand side of equation (4) is enclosing given volume and is not the open surface. The Stoke's theorem applied to closed surface produces zero answer.

$$\therefore \quad \int_{vol} \alpha \; dv = 0 \qquad \ldots (5)$$

This is true for differential volume also.

$$\therefore \quad \alpha \, dv = 0 \qquad \ldots (6)$$

But $\quad dv \neq 0 \quad$ as it is a differential volume.

$$\therefore \quad \alpha = 0 \qquad \ldots (7)$$

From equation (1),

$$\nabla \cdot (\nabla \times \bar{A}) = 0 \qquad \ldots (8)$$

This proves that the divergence of curl of a vector is zero.

2.10 Laplacian of a Scalar

A single operator which is the composite of gradient and divergence operator known as the Laplacian of a scalar.

If V is a scalar field, then the Laplacian of a scalar V is denoted as $\nabla^2 V$ and mathematically defined as the divergence of the gradient of V.

In Cartesian coordinate system it is given by,

$$\nabla^2 V = \nabla . \nabla \mathrm{V} = \left[\frac{\partial}{\partial x}\overrightarrow{a_x} + \frac{\partial}{\partial y}\overrightarrow{a_y} + \frac{\partial}{\partial z}\overrightarrow{a_z}\right] . \left[\frac{\partial V}{\partial x}\overrightarrow{a_x} + \frac{\partial V}{\partial y}\overrightarrow{a_y} + \frac{\partial V}{\partial z}\overrightarrow{a_z}\right]$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \ldots\ldots (1)$$

Key Point: The Laplacian of a scalar is always a scalar.

In Cylindrical coordinate system, it is given as,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \ldots\ldots (2)$$

In Spherical coordinate system, it is given as,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \ldots\ldots (3)$$

Harmonic Field: A scalar field is said to be harmonic in a given region, if its Laplacian vanishes in that region.

Mathematically for a scalar field V to be **harmonic**,

$$\nabla^2 V = 0 \ldots\ldots (4)$$

This equation is called **Laplace's equation**.

Example 1.22 : *Find the Laplacian of the scalar fields and comment on, which fields are harmonic.*

i) $W = x^2 y + xyz - yz^2$ *ii)* $U = rz \sin\phi + z^2 \cos^2\phi + r^2$

iii) $V = 2r \cos\theta \cos\phi$

Solution : i) $W = x^2y + xyz - yz^2$

$$\therefore \quad \nabla^2 W = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz - z^2) + \frac{\partial}{\partial z}(xy - 2yz)$$

$$= 2y + 0 + 0 + 0 + 0 - 2y = 0$$

As $\nabla^2 W = 0$, the scalar field W is **harmonic.**

ii) $U = rz \sin\phi + z^2 \cos^2\phi + r^2$

$$\therefore \quad \nabla^2 U = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}[r(z\sin\phi + 2r)] + \frac{1}{r^2}\frac{\partial}{\partial \phi}[rz\cos\phi - 2z^2 \sin\phi\cos\phi] + \frac{\partial}{\partial z}[r\sin\phi + 2z\cos^2\phi]$$

$$\ldots\ 2\sin\phi\cos\phi = \sin 2\phi$$

$$= \frac{1}{r}[z\sin\phi + 4r] + \frac{1}{r^2}[-rz\sin\phi - z^2 2\cos 2\phi] + [0 + 2\cos^2\phi]$$

$$= \frac{z}{r}\sin\phi + 4 - \frac{z}{r}\sin\phi - \frac{2z^2}{r^2}\cos 2\phi + 2\cos^2\phi$$

$$= 4 + 2\cos^2\phi - \frac{2z^2}{r^2}\cos 2\phi$$

As $\nabla^2 U \neq 0$, the scalar field U is **not harmonic.**

iii) $V = 2r\cos\theta\cos\phi$

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}[r^2(2\cos\theta\cos\phi] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}[\sin\theta(-2r\cos\phi\sin\theta)] + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}(-2r\cos\theta\sin\phi)$$

$$= \frac{1}{r^2}[4r\cos\theta\cos\phi] + \frac{1}{r^2\sin\theta}[-2r\cos\phi \times 2\sin\theta\cos\theta] + \frac{1}{r^2\sin^2\theta}[-2r\cos\theta\cos\phi]$$

$$= \frac{4\cos\theta\cos\phi}{r} - \frac{4\cos\theta\cos\phi}{r} - \frac{2\cos\theta\cos\phi}{r\sin^2\theta} = \frac{-2}{r}\cot\theta\,\mathrm{cosec}\,\theta\cos\phi$$

As $\nabla^2 V \neq 0$, the scalar field V is **not harmonic.**

Review Questions

1. What is a scalar and scalar field ? Give two examples.
2. What is a vector and vector field ? Give two examples.
3. What is a unit vector ? What is its significance in the vector representation ? How to find unit vector along a particular vector ?
4. Explain cartesian co-ordinate system and differential elements in cartesian co-ordinate system.
5. Explain cylindrical co-ordinate system and differential elements in cylindrical co-ordinate system.
6. Explain spherical co-ordinate system and differential elements in spherical co-ordinate system.
7. What is a dot product ? Explain its significance and applications.
8. What is a cross product ? Explain its properties and applications.
9. Explain the relationship between cartesian and cylindrical as well as cartesian and spherical systems.
10. How to transform the vectors from one coordinate system to other ?

11. Given two points A (5, 4, 3) and B (2, 3, 4).
    Find : i) $\overline{A}+\overline{B}$ ii) $\overline{A}\bullet\overline{B}$ iii) θ_{AB} iv) $\overline{A}\times\overline{B}$
    v) Unit vector normal to the plane containing $\overline{A}$ and $\overline{B}$.
    vi) Area of parallelogram of which $\overline{A}$ and $\overline{B}$ are adjacent sides.
    [Ans. : $7\bar{a}_x + 7\bar{a}_y + 7\bar{a}_z$, 34, 26.762°, $0.41\,\bar{a}_x - 0.82\,\bar{a}_y + 0.41\,\bar{a}_z$, 17.1464]
    [Hint. : For area $|\overline{A}|\,|\overline{B}|\sin\theta_{AB} = |\overline{A}\times\overline{B}|$]

12. If two positions vectors given are, $\overline{A} = -2\,\bar{a}_x - 5\,\bar{a}_y - 4\,\bar{a}_z$ and $\overline{B} = 2\,\bar{a}_x + 3\,\bar{a}_y + 5\,\bar{a}_z$ then find,
    i) $\overline{AB}$ ii) $\bar{a}_A$ iii) $\bar{a}_B$ iv) $\bar{a}_{AB}$ v) Unit vector in the direction from C to A where C is (3, 5, 8).
    [Ans. : $4\bar{a}_x + 8\bar{a}_y + 9\bar{a}_z$, $-0.298\,\bar{a}_x - 0.745\,\bar{a}_y - 0.596\,\bar{a}_z$, $-0.324\,\bar{a}_x + 0.486\,\bar{a}_y - 0.811\,\bar{a}_z$, $0.315\,\bar{a}_x + 0.63\,\bar{a}_y + 0.71\,\bar{a}_z$, $-0.304\,\bar{a}_x - 0.61\,\bar{a}_y - 0.732\,\bar{a}_z$]
13. Find the value of B_z such that the angle between the vectors $\overline{A} = 2\,\bar{a}_x + \bar{a}_y + 4\,\bar{a}_z$ and $\overline{B} = -2\,\bar{a}_x - \bar{a}_y + B_z\,\bar{a}_z$ is 45°. **[Ans. : 7.9]**
14. For the vectors, $\overline{A} = 2\,\bar{a}_x - 2\,\bar{a}_y + \bar{a}_z$ and $\overline{B} = 3\,\bar{a}_x + 5\,\bar{a}_y - 2\,\bar{a}_z$ find $\overline{A}\bullet\overline{B}$, $\overline{A}\times\overline{B}$ and show that $\overline{A}\times\overline{B} = -\left(\overline{B}\times\overline{A}\right)$.
    [Ans. : -6, $-\bar{a}_x + 7\,\bar{a}_y + 16\bar{a}_z$]
15. Show that $\overline{A} = 4\,\bar{a}_x - 2\,\bar{a}_y - \bar{a}_z$ and $\overline{B} = \bar{a}_x + 4\,\bar{a}_y - 4\,\bar{a}_z$ are mutually perpendicular vectors.
    [Hint. : Show $\overline{A}\bullet\overline{B} = 0$]
16. Find the angle between the vectors, $\overline{A} = 2\,\bar{a}_x + 4\,\bar{a}_y - \bar{a}_z$ and $\overline{B} = 3\,\bar{a}_x + 6\,\bar{a}_y - 4\,\bar{a}_z$ using dot product and cross product. **[Ans. : 18.21°]**
17. Consider two vectors $\overline{P} = 4\bar{a}_y + 10\bar{a}_z$ and $\overline{Q} = 2\bar{a}_x + 3\,\bar{a}_y$. Find the projection of $\overline{P}$ and $\overline{Q}$.
    [Ans. : 3.328]
18. Given the points A (x = 2, y = 3, z = − 1) and $B(r = 4, \Phi = -50°. z = 2)$, find the distance of A and B from the origin. Also find distance A to B. **[Ans. : 3.74, 4.47, 6.78]**
19. Given the two points A (x = 2, y = 3, z = − 1) and $B(r = 4, \theta = 25°, \phi = 120°)$. Find the spherical coordinates of A, cartesian coordinates of B and distance AB.
    [Ans. : A (3.74, 105.5°, 56.31°), B (− 0.845, 1.46, 3.627, 5.64)]
20. Transform the vector $5\,\bar{a}_x$ at Q (x = 3, y = 4, z = − 2) to the cylindrical co-ordinates.
    [Ans. : $3\,\bar{a}_r - 4\,\bar{a}_\phi$]
21. What is Laplacian of a scalar field ? What is its significance.
22. Find the Laplacian of the following scalar fields :
    i) $W = e^{-z}\sin 2x \cosh y$ **[Ans. : $-2\,e^{-z}\sin 2x \cosh y$]**
    ii) $V = 10\, r \sin^2\theta\cos\phi$ **[Ans. : $\dfrac{10\cos\phi}{r}(1 + 2\cos 2\theta)$]**

University Questions

1. *What do you mean by Scalar and Vector Fields ? Show the difference between the two.* **[UPTU : 2002-03, 5 Marks]**
2. *Give the physical interpretation of gradiant and curl of a vector.* **[UPTU : 2003-04(A), 5 Marks]**
3. *Represent the dot product of ∇ with vector field in spheircal co-ordinate system.* **[UPTU : 2003-04(B), 5 Marks]**
4. *Give the physical interpretation of gradient, divergence and curl of a vector field.* **[UPTU : 2003-04(B), 5 Marks]**
5. *Discuss the Stokes' theorem and its application.* **[UPTU : 2003-04(B), 5 Marks]**
6. *Verify that vector field* $\overline{A} = yz\overline{a}_x + zx\overline{a}_y + xy\overline{a}_z$ *is irrotational and solenoidal.* **[UPTU : 2005-06, 5 Marks]**
7. *Write down gradient of any scalar and divergence and curl of any vector.* $\vec{A}$ *in different co-ordiante system.* **[UPTU : 2006-07, 5 Marks]**
8. *If* $\overline{A} = \alpha\,\overline{a}_x + 2\,\overline{a}_y + 10\,\overline{a}_z$ *and* $\overline{B} = 4\alpha\,\overline{a}_x + 8\,\overline{a}_y - 2\alpha\,\overline{a}_z$*, find out the value of* α *for which the two vectors become perpendicular.* **[UPTU : 2006-07, 5 Marks]**
9. *Given ponts* $A(x = 2, y = 3, z = -1)$ *and* $B(\rho = 4, \phi = -50°, z = 2)$ *find the distance A to B.* **[UPTU : 2006-07, 5 Marks]**
10. *Show that the vector fields*

$$\overline{A} = \overline{a}_r \frac{\sin 2\theta}{r^2} + 2\overline{a}_\theta \frac{(\sin\theta)}{r^2} \text{ and}$$

$\overline{B} = r\cos\theta\overline{a}_r + r\overline{a}_\theta$ *are every where parallel to each other.* **[UPTU : 2007-08, 5 Marks]**

11. *Express the field* $\overline{E} = \frac{A}{r^2}\overline{a}_r$ *in (i) rectangular components, ii) cylindrical components.* **[UPTU : 2007-08, 5 Marks]**
12. *Write down the word statement of divergence theorem and Stokes theorem. Find out the divergence and curl of the following function.*

$\overline{A} = 2xy\,\overline{a}_x + x^2z\overline{a}_y + z^3\overline{a}_z$ **[UPTU : 2007-08, 5 Marks]**

13. *Establish the following vector identities :*

i) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$

ii) $\nabla \cdot (\nabla \times \vec{A}) = 0$ **[UPTU : 2008-09, 10 Marks]**

14. *Discuss the following terms as applied to vector fields :*

i) Gradient

ii) Divergence

iii) Curl and its physical interpretation **[UPTU : 2008-09, 10 Marks]**